

K-THEORY OF SEMI-LINEAR ENDOMORPHISMS VIA THE RIEMANN–HILBERT CORRESPONDENCE

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ABSTRACT. Grayson, developing ideas of Quillen, has made computations of the K -theory of ‘semi-linear endomorphisms’. In the present text we develop a technique to compute these groups in the case of Frobenius semi-linear actions. The main idea is to interpret the semi-linear modules as crystals and use a positive characteristic version of the Riemann–Hilbert correspondence. We also compute the K -theory of the category of étale constructible p -torsion sheaves.

If R is a commutative regular \mathbf{F}_q -algebra, we may define the Frobenius skew ring

$$(0.1) \quad R[F] := R\{F\} / \langle x^q F - Fx \mid x \in R \rangle.$$

This is a non-commutative ring. If we ignore the meaning of $q := \#\mathbf{F}_q$ and pretended “ $q = 1$ ” formally, this definition would output the ordinary polynomial ring. However, one of the most fundamental properties of K -theory is its \mathbf{A}^1 -invariance, i.e.

$$(0.2) \quad K(R) \xrightarrow{\sim} K(R[T]),$$

so one is tempted to hope that this remains true when $q \neq 1$. Our first result is that this is indeed the case:

Theorem. *Let X/\mathbf{F}_q be a smooth separated scheme. Then there is an equivalence in K -theory*

$$K(X) \xrightarrow{\sim} K(\mathrm{Coh}_{\mathcal{O}_X[F]}(X)).$$

Here $\mathrm{Coh}_{\mathcal{O}_X[F]}(X)$ refers to coherent right $\mathcal{O}_X[F]$ -module sheaves.

See Theorem 3.15. The proof will be much like the one for “ $q = 1$ ”, but with a critical technical complication: $\mathcal{O}_X[F]$ is neither left nor right Noetherian when $q \neq 1$. But this can be managed, thanks to Emerton’s insight that they remain coherent rings [Eme08], i.e. all finitely generated ideals are automatically finitely presented. He shows left coherence, and we complement this with right coherence in the present text.

Alternatively, we could look at those (right) $\mathcal{O}_X[F]$ -module sheaves which are additionally coherent as \mathcal{O}_X -module sheaves. This yields the category of coherent Cartier modules $\mathrm{CohCart}(X)$ of Blickle and Bökke [BB11]. Its K -theory sees additional arithmetic information:

Theorem. *Suppose X/\mathbf{F}_q is a smooth separated scheme. Then there is a long exact sequence in K -theory*

$$\cdots \rightarrow K_m(X) \rightarrow K_m(\mathrm{CohCart}(X)) \rightarrow K_m(\acute{\mathrm{E}}t_c(X, \mathbf{F}_q)) \rightarrow \cdots.$$

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Here $\dot{\mathcal{E}}t_c(X, \mathbf{F}_q)$ denotes the abelian category of constructible étale sheaves with \mathbf{F}_q -coefficients. This computation is based on the positive characteristic version of the Riemann–Hilbert correspondence of Emerton–Kisin [EK04b] and Blickle–Böckle [BB11].

We also describe $K(\dot{\mathcal{E}}t_c(X, \mathbf{F}_q))$ to some extent. So far, the only computation regarding the K -theory of this category that I’ve seen in the literature is due to Taelman [Tae15]. He develops a function-sheaf correspondence for these étale sheaves. To each sheaf and \mathbf{F}_q -rational point, one pulls back the sheaf to this point and takes the trace of the Frobenius action. This assigns a value to each rational point. Taelman shows that this construction factors over the K_0 -group of $\dot{\mathcal{E}}t_c(X, \mathbf{F}_q)$, and thus gives rise to a short exact sequence

$$(0.3) \quad 0 \rightarrow K_0(\dot{\mathcal{E}}t_c(X, \mathbf{F}_q))^{\mathrm{Tr}=0} \rightarrow K_0(\dot{\mathcal{E}}t_c(X, \mathbf{F}_q)) \rightarrow \bigoplus_{x \in X(\mathbf{F}_q)} \mathbf{F}_q \rightarrow 0$$

and he gives explicit generators for the term on the left-hand side. We augment this computation of the K_0 -group as follows:

Theorem. *Suppose X/\mathbf{F}_q is a smooth scheme. Then*

$$K_m(\dot{\mathcal{E}}t_c(X, \mathbf{F}_q)) = \begin{cases} \text{prime-to-}p \text{ torsion} & \text{for } m = 2i + 1 > 0, \\ 0 & \text{for } m = 2i, i > 0, \\ \bigoplus \mathbf{Z} & \text{for } m = 0, \end{cases}$$

where the direct sum in the last row runs over all simple Cartier crystals on X (or equivalently simple perverse étale \mathbf{F}_q -sheaves). Among them, there is a canonical set of $\#X(\mathbf{F}_q)$ generators which surject on the right-hand side in Sequence 0.3, while the remaining generators all map to zero under the function-sheaf correspondence.

See Theorem 4.13. For this result, we first switch to the perverse t -structure by Neeman’s theorem of the heart, and then to Cartier crystals under the Riemann–Hilbert correspondence and work in the latter context. There, the computation crucially uses the strong finiteness properties proven by Blickle and Böckle [BB11], and Quillen’s computation of the K -theory of finite fields.

There is also a completely different angle from which to look at this text: Instead of the K -theory of a category itself, one may look at the category of pairs (X, α) , where X is an object and α acts on it via (1) an endomorphism, (2) or an automorphism, or (3) a semi-linear endomorphism of X . Grayson has a series of articles investigating these cases [Gra77], [Gra79], [Gra88]. The case of automorphisms has gained some fame for its rôle in the motivic Atiyah–Hirzebruch spectral sequence [Gra95], and recently there has been some renewed interest in the cases of endo- and automorphisms, e.g. [Tab14], [BGT16].

The case of semi-linear actions seems to be studied less. Basically, an example of Quillen in [Qui73] and Grayson’s paper [Gra88] seem to be the only ones computing higher K -groups (beyond K_1) in the case of semi-linear actions. Seen from this angle, the above theorems provide a large supply of further computations in the special case when the semi-linear action comes from the Frobenius. Grayson’s paper used a “Frobenius \mathbf{P}^1 ”, which inspires our use of a twisted affine line, however the projective line seems to be definable only for rings for which the Frobenius is an automorphism.

In this direction, the technical advance of the present text is that we can handle rings where the Frobenius is *not* surjective. This is the key point which allows us to handle rings and varieties over \mathbf{F}_q of dimension ≥ 1 .

Further results:

This paper has a precursor in a computation in Quillen's "Higher Algebraic K -Theory I". Quillen considers the K -theory of semi-linear endomorphisms of the single point $X := \text{Spec}(\mathbf{F}_p^{\text{sep}})$. Based on it, he defines a skew field D and computes its K -theory. The meaning of these K -theory classes remained mysterious. We try to elucidate his computation by adapting it to schemes over \mathbf{F}_p in §5. We explain that it is impossible to define an analogous skew field in this generality, but there is an abelian category "QD", which imitates the behaviour of modules over the non-existing D . Assuming Parshin's conjecture, we can compute its rational K -theory and find

Quillen's $K_m(D)_{\mathbf{Q}}$	Generalized $K_m(\text{QD}(X))_{\mathbf{Q}}$	m
\mathbf{Q}	\mathbf{Q}	0
$\mathbf{Q} \oplus \mathbf{Q}$	$\mathbf{Q} \oplus K_0(\dot{\text{Et}}_c(X, \mathbf{F}_q))_{\mathbf{Q}}$	1
0	0	≥ 2 .
for $X = \text{Spec}(\mathbf{F}_p^{\text{sep}})$	for X smooth, projective over \mathbf{F}_q	

One can see how the individual summands in Quillen's computation generalize, which might be a first step in understanding the bigger picture behind Quillen's computation.

1. PREPARATIONS

Conventions: A ring R denotes an associative unital algebra, not necessarily commutative. Ring morphisms are always supposed to preserve the unit. If R is a ring, $\text{Mod}_{fg}(R)$ resp. $\text{Mod}_{fp}(R)$ denote the category of finitely generated resp. finitely presented right R -modules. We write $\text{P}_f(R)$ to denote the exact category of finitely generated projective right R -modules. We call a ring *right regular* if every finitely presented right module has finite projective dimension. Unlike most commutative algebra texts, we do not demand regular rings to be Noetherian. In particular, left and right global dimensions may differ.

We pick once and for all a prime number p and a prime power $q = p^r$ with $r \geq 1$. Let R be a ring. We write $R\{X\}$ for the free associative ring in a non-commuting variable X . We write $R[X]$ to denote the polynomial ring over R , i.e. in this case the variable X commutes with all elements of the ring. In other words,

$$(1.1) \quad R[X] = R\{X\} / \langle rX - Xr \mid r \in R \rangle.$$

We will only deviate from this notation in one special case: Suppose R denotes a commutative \mathbf{F}_q -algebra. Define the non-commutative *Frobenius skew ring*

$$(1.2) \quad R[F] := \frac{R\{F\}}{\langle x^q F - Fx \mid x \in R \rangle}.$$

So we reserve the special letter " F " for this definition differing from the one in line 1.1. This is fairly common practice in the literature.

1.1. Strategy. Before we start with precise arguments, let us just explain what we want to do: The ring inclusion $R \subset R[F]$ induces a morphism in K -theory

$$K(R) \rightarrow K(R[F])$$

and we would like to show that this functor induces an equivalence in K -theory. The idea is to imitate Quillen's proof of \mathbf{A}^1 -invariance of K -theory for regular R : He proves the equivalence $K(R) \xrightarrow{\sim} K(R[T])$, where $R[T]$ is the ordinary polynomial ring and R a regular Noetherian ring. The key tool in Quillen's proof is the following result:

Theorem 1.1 ([Qui73, Theorem 7]). *Let A be an increasingly filtered ring*

$$A = \bigcup_{s \geq 0} A^{\leq s}$$

such that $A^{\leq s} \cdot A^{\leq t} \subseteq A^{\leq s+t}$. Suppose the associated graded $\mathrm{Gr} A := \bigoplus_{s \geq 0} A^{\leq s} / A^{\leq s-1}$ (with the tacit understanding that $A^{\leq -1} := 0$) is right Noetherian and has finite Tor-dimension as a right module over $A^{\leq 0}$. Moreover, assume that $A^{\leq 0}$ has finite Tor-dimension as a right $\mathrm{Gr} A$ -module. Then the ring homomorphism $A^{\leq 0} \hookrightarrow A$ induces an equivalence in K -theory

$$K(\mathrm{Mod}_{fg} A^{\leq 0}) \xrightarrow{\sim} K(\mathrm{Mod}_{fg} A).$$

Quillen then combines this with a comparison result between the K -theory of finitely generated projective right A -modules and finitely generated right A -modules, namely

$$(1.3) \quad K(A^{\leq 0}) \xrightarrow{\sim} K(\mathrm{Mod}_{fg} A^{\leq 0}) \quad \text{and} \quad K(A) \xrightarrow{\sim} K(\mathrm{Mod}_{fg} A),$$

which requires A_0 and A to be right Noetherian and right regular. So, this is the plan. However, we cannot just follow this strategy, because it collapses at a number of places for the Frobenius skew ring $R[F]$:

- (1) The ring $R[F]$ is practically never right Noetherian. In particular, the category of finitely generated right $R[F]$ -modules, $\mathrm{Mod}_{fg} R[F]$, a priori need not be an abelian category. For the moment, this is not too bad, as it still is an exact category and thus has a notion of K -theory.
- (2) The ring $R[F]$ is indeed filtered by $R[F]^{\leq d} := \{\sum_{i=0}^d r_i F^i\}$. One easily computes that

$$\mathrm{Gr} R[F] \simeq R[F],$$

i.e. the associated graded is isomorphic to the original ring. However, since $R[F]$ is rarely right Noetherian, this means that $\mathrm{Gr} R[F]$ will fail to be right Noetherian, too. So it cannot satisfy the assumptions of Theorem 1.1.

- (3) The analogues of the comparison results in line 1.3 require right regularity. This is in fact a rather recent result of Linquan Ma [Ma14, Theorem 3.2].
- (4) We solve the non-Noetherian problem by proving right coherence of $R[F]$ under suitable conditions (based on a method of Emerton, [Eme08]). But this still does not quite suffice because if a ring A is right coherent, its polynomial ring $A[T]$ need not be right coherent as well (by an example due to Soublin [Sou70, §5]), and we will have no better tool than proving the relevant right coherence statements by hand. Based on this, we can then use a strengthening of Quillen's theorem, Theorem 1.1, for right coherent rings, due to Gersten [Ger74].

2. RING-THEORETIC PROPERTIES OF THE FROBENIUS SKEW RING

2.1. Generalities. Let us collect a few properties of the Frobenius skew ring, defined as in line 1.2: Suppose R is a commutative \mathbf{F}_q -algebra. Every element in $R[F]$ has a unique presentation as a left polynomial

$$\alpha = \sum_{i=0}^d r_i F^i \quad \text{with} \quad r_i \in R.$$

Thus, $R[F]$ is a free left R -module. It is also a right R -module because of $rF \cdot s = rs^q F$. However, it need not be free as a right module, nor will it in general be possible to represent elements as right polynomials $\sum F^i r_i$.

If $f : R \rightarrow S$ is a ring morphism of commutative \mathbf{F}_q -algebras, the defining relation in line 1.2 is preserved and one obtains an induced morphism $R[F] \rightarrow S[F]$.

We would like to speak about finitely generated left or right $R[F]$ -modules. However, we directly run into problems since $R[F]$ is only very rarely left or right Noetherian. This is a problem because for a general ring, its category of finitely generated modules will not even be an abelian category.

The question of being Noetherian was settled in full generality by Yuji Yoshino [Yos94].

Theorem 2.1 (Yoshino). *Suppose R is a Noetherian commutative \mathbf{F}_p -algebra.*

- (1) *Then $R[F]$ is left Noetherian iff R is a direct product of finitely many fields.*
- (2) *Then $R[F]$ is right Noetherian iff R is Artinian and all closed points in $\text{Spec } R$ have perfect residue fields.*

See [Yos94, Theorem 1.3]. The proof of this general version is quite involved. In the classical case of $R = k$ a perfect field, one can interpret $R[F]$ as a twisted polynomial ring. This case has textbook treatments, e.g. [MR01, 2.9, Theorem, (iv)].

As we can see, we need a workaround handling the lack of Noetherian properties since the above cases are far too special to be useful. They all have $\text{Spec } R$ zero-dimensional. We shall use the formalism of coherent rings. We recall all necessary foundations:

Suppose A is a ring. A right A -module is called *coherent* if (1) it is finitely generated, and (2) every finitely generated right submodule is finitely presented. For every short exact sequence of right A -modules,

$$0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$$

all modules are coherent as soon as any two of them are coherent (see Soublin's survey [Sou70] for this and related properties).

A ring A is called *right coherent* if every finitely generated right ideal is also finitely presented. Equivalently, every finitely generated right submodule of a free right module is finitely presented. Clearly right Noetherian rings are also right coherent.

Example 2.2 ([Sou70, §5, Corollaire]). The simplest example of a ring which is not right Noetherian, but right coherent, is the polynomial ring in countably many variables over a commutative Noetherian ring R , i.e. $A := R[X_1, X_2, \dots]$.

Proposition 2.3. *For a ring A , the following are equivalent:*

- (1) *A is right coherent.*
- (2) *A is a coherent right A -module over itself.*
- (3) *The category of finitely presented right A -modules is abelian.*

We shall need the following facts:

Proposition 2.4. *Suppose A is a right coherent ring.*

- (1) *Then every finitely generated projective right A -module is coherent.* ([Sou70, §3, Prop. 9])
- (2) *If I is a two-sided ideal in A , which is finitely generated as a right A -module, then the quotient ring A/I is also right coherent.* ([Sou70, §4, Cor. 1])

The latter fact implies that if the polynomial ring $A[T]$ is right coherent, so is A . The converse direction is known to be false:

Example 2.5 (Soublin). There exists a commutative coherent ring A such that $A[T]$ is not coherent. See [Sou70, §5, Prop. 18].

2.2. Right stable coherence. So, the notion of a coherent ring is not \mathbf{A}^1 -invariant. Inspired by this, Gersten has introduced the following notion:

Definition 2.6 ([Ger74, Definition 1.2]). *A ring A is called right stably coherent (a.k.a. “right super-coherent”) if for any index set I the multi-variable polynomial ring $A[X_i]_{i \in I}$ is right coherent.¹*

Now, Matt Emerton has shown that the Frobenius skew ring is left coherent.

Theorem 2.7 (Emerton [Eme08]). *Suppose*

- *R is a Noetherian commutative \mathbf{F}_q -algebra, and*
- *R is F -flat (e.g. if R is regular²).*

Then the ring $R[F]$ is left coherent.

This is the main result of [Eme08]. We will now prove the right analogue of Emerton’s theorem. The assumption of F -flatness will need to be replaced by F -finiteness, as it turns out. Apart from the minor modifications circling around this, the proof will be a mirror image of Emerton’s proof, albeit augmented with additional commuting polynomial variables.

Because only right stable coherence will be useful for us later, we shall need to prove right coherence for polynomial rings over $R[F]$. Let us set up the notation. We have

$$(2.1) \quad R[F][T] := \left\{ \sum r_{i,\ell} F^i T^\ell \mid r_{i,\ell} \in R, \text{ all but finitely many } r_{i,\ell} \text{ are zero} \right\}$$

and recall that by the construction of this ring, we have the relations

$$(2.2) \quad r F^i T^\ell \cdot s = r s^{q^i} F^i T^\ell \quad T \cdot r F^i T^\ell = r F^i T^{\ell+1} \quad T F = F T$$

for all $r, s \in R$. In particular, T commutes with all other terms.

Example 2.8. The ring $R[F][T]$ is different from $R[T][F]$, because the latter ring satisfies the relation $FT = T^q F$ instead.

¹Gersten has called such rings ‘right super-coherent’. However, the majority of the literature prefers to call them ‘right stably coherent’. Indeed, Aschenbrenner has introduced another concept called super-coherence, which is related, but different from stable coherence, and in particular different from Gersten’s usage.

²By Kunz Theorem [Kun69, Theorem 2.1], if R is a reduced Noetherian commutative \mathbf{F}_q -algebra, it is F -flat if and only if it is regular.

Remark 2.9. All of the following considerations also work verbatim for multi-variable polynomial rings over $R[F]$, i.e. $R[F][T_1, \dots, T_m]$ for any integer $m \geq 0$. In fact, just read the definition in line 2.1 as allowing a multi-index $\underline{\ell} = (\ell_1, \dots, \ell_m)$ for ℓ , and the same finiteness condition on the coefficients $r_{i,\ell}$. For the sake of legibility, we proceed by writing a single T .

Definition 2.10 (Right twisting the action). *If M is a right $R[T]$ -module, write \widetilde{M} for the right $R[T]$ -module with the same elements as M , but with the twisted right $R[T]$ -module structure*

$$m \cdot_{\widetilde{M}} sT^j := m \cdot_M s^q T^j.$$

When convenient, we may even use this notation for right $R[F][T]$ -modules, meaning

$$m \cdot_{\widetilde{M}} sF^i T^j := m \cdot_M s^q F^i T^j.$$

If there is also a left $R[T]$ -module structure on M , we do not change it.

Remark 2.11. One can also denote this by $M^{(1)}$ or by F_*M for $F : R \rightarrow R$ being the map $r \mapsto r^q$. The latter might actually be the most official notation, but I am very hesitant to use it. Firstly, if M carries both a left and right F action, the notation F_*M is very ambiguous. Secondly, we will switch a lot between schemes and rings and then the Frobenius morphism goes in opposite directions depending on which viewpoint we use. Both feels all too prone to lead to confusion.

Lemma 2.12. *Suppose R is a commutative \mathbf{F}_q -algebra. The following are equivalent:*

- (1) *R is a finitely generated (right) R^q -module.*
- (2) *\widetilde{R} is a finitely generated (right) R -module.*
- (3) *For every $d \geq 0$, R is a finitely generated (right) R^{q^d} -module.*

The proof is straightforward. The ring R is called *F-finite* if these conditions are met. This property is very common. Perfect fields are *F-finite*, and *F-finiteness* is closed under taking finitely generated algebra extensions, localizations and homomorphic images.

Lemma 2.13. *Suppose R is a commutative F -finite \mathbf{F}_q -algebra.*

- (1) *If M is a finitely generated right $R[F][T]$ -module, then so is \widetilde{M} .*
- (2) *If R is additionally Noetherian: If M is a finitely presented right $R[F][T]$ -module, then so is \widetilde{M} .*

Proof. This follows literally from F being a finite morphism. (1) Concretely, for finite generation, if $\{b_i\}_{i \in I}$ is a finite set of right generators for M , and $\{\rho_s\}_{s \in J}$ a finite set of generators of R as a right R^q -module, then every element $m \in \widetilde{M}$ can be written as a finite sum $m = \sum b_i f_i$ for suitable $f_i \in R[F][T]$, and each of these as $f_i = \sum_{j,k} r_{i,j,k} F^j T^k = \sum_{j,k,s} \rho_s r_{i,j,k,s}^q F^j T^k$. Hence,

$$m = \sum_{i,s} b_i \rho_s \cdot_M \sum_{j,k} r_{i,j,k,s}^q F^j T^k = \sum_{i,s} b_i \rho_s \cdot_{\widetilde{M}} \sum_{j,k} r_{i,j,k,s} F^j T^k,$$

proving that $\{b_i \rho_s\}_{i \in I, s \in J}$ is a finite set of generators as a right $R[F][T]$ -module. (2) As \widetilde{R} is a finitely generated R -module, and R is Noetherian, \widetilde{R} is also a finitely presented

R -module, i.e. there are n, m such that $R^{\oplus n} \rightarrow R^{\oplus m} \rightarrow \tilde{R} \rightarrow 0$ is exact. Since $R[F][T]$ is a free (thus flat) left R -module, tensoring it from the right yields

$$R[F][T]^{\oplus n} \rightarrow R[F][T]^{\oplus m} \rightarrow \tilde{R} \otimes_R R[F][T] \rightarrow 0$$

and the term on the right equals $\widetilde{R[F][T]} = \widetilde{R[F][T]}$ (recall that by our definition of $\widetilde{(-)}$, only the right R -module structure changes, e.g. right multiplication by r becomes right multiplication by r^q , but right multiplication by T remains right multiplication by T). Thus, $\widetilde{R[F][T]}$ is a finitely presented right $R[F][T]$ -module. Hence, if

$$R[F][T]^{\oplus n} \rightarrow R[F][T]^{\oplus m} \rightarrow M \rightarrow 0$$

is a finite presentation of a right $R[F][T]$ -module M , we get

$$\widetilde{R[F][T]}^{\oplus n} \rightarrow \widetilde{R[F][T]}^{\oplus m} \rightarrow \widetilde{M} \rightarrow 0,$$

presenting \widetilde{M} as a quotient of a finitely presented module by a finitely generated one, implying that it is finitely presented itself. \square

Corollary 2.14. *Suppose R is a commutative F -finite Noetherian \mathbf{F}_q -algebra. If M is a finitely generated (resp. presented) R -module, then so is \widetilde{M} .*

For a short exact sequence of right $R[F][T]$ -modules, one gets the diagram of right R -modules

$$\begin{array}{ccccccc} 0 & \rightarrow & \tilde{A} & \rightarrow & \tilde{B} & \rightarrow & \tilde{C} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C \rightarrow 0. \end{array} \quad \cdot F$$

The downward arrows denote the *right* action by F . Thanks to the tilde twist in the upper row, this is a right R -module homomorphism (had we not twisted the right action in the top row, this would only be an abelian group homomorphism).

The following lemmata are right analogues of corresponding statements in [Eme08], suitably stabilized along additional commuting polynomial variables.

Lemma 2.15. *Suppose R is a commutative F -finite \mathbf{F}_q -algebra. Then for all $d \geq 0$ the following set is a finitely generated (right) $R[T]$ -module:*

$$Z^{\leq d} := \left\{ \sum_{i=0}^d \sum_{\ell} r_{i,\ell} F^i T^\ell \mid r_{i,\ell} \in R, \text{ all but finitely many } r_{i,\ell} \text{ are zero} \right\}$$

Proof. First of all, we see that it is a right $R[T]$ -module by

$$r_{i,\ell} F^i T^\ell \cdot s = r_{i,\ell} s^{q^i} F^i T^\ell \quad \text{and} \quad r_{i,\ell} F^i T^\ell \cdot T = r_{i,\ell} F^i T^{\ell+1}.$$

By the previous lemma and our assumption that R be F -finite, R is a finitely generated (right) R^{q^d} -module. Thus, every element $r \in R$ can be written in the shape $r = \sum_{s \in \mathcal{B}_d} s \rho_s^{q^d}$, where \mathcal{B}_d is a finite set of right generators for R and ρ_s the coefficients as a right R^{q^d} -module (i.e. they act as $\rho_s^{q^d}$ with respect to the ordinary right R -module structure). Hence, every element in $Z^{\leq d}$ has the shape

$$z = \sum_{\ell} \sum_{i=0}^d r_{i,\ell} F^i T^\ell = \sum_{\ell} \sum_i \left(\sum_{s \in \mathcal{B}_i} s \rho_{i,\ell,s}^{q^i} \right) F^i T^\ell$$

and this equals

$$= \sum_{i=0}^d \sum_{s \in \mathcal{B}_i} s F^i \sum_{\ell} \rho_{i,\ell,s} T^\ell.$$

We see that $Z^{\leq d}$ is spanned as a right $R[T]$ -module by $\langle s F^i \rangle_{0 \leq i \leq d, s \in \mathcal{B}_i}$. This is a finite set since each \mathcal{B}_i is finite and i runs through finitely many values only. \square

Lemma 2.16. *Suppose R is a Noetherian F -finite commutative \mathbf{F}_q -algebra. Then for every right ideal I in $R[F][T]$, each*

$$I^{\leq d} := I \cap Z^{\leq d}$$

is a finitely generated (right) $R[T]$ -submodule of I and their union $\bigcup_{d \geq 0} I^{\leq d}$ is all of I .

Proof. For any $d \geq 0$, we have

$$I^{\leq d} = \left\{ \sum_{\ell} \sum_{i=0}^d r_{i,\ell} F^i T^\ell \mid r_{i,\ell} \in R, \text{ all but finitely many } r_{i,\ell} \text{ are zero} \right\} \cap I.$$

Since both I and $Z^{\leq d}$ are right $R[T]$ -modules, so is their intersection $I^{\leq d}$. By Lemma 2.15 the right module $Z^{\leq d}$ is a finitely generated (right) $R[T]$ -module. Since R is Noetherian, the (commutative) polynomial ring $R[T]$ is Noetherian. It follows that $Z^{\leq d}$ is a Noetherian right $R[T]$ -module and thus its $R[T]$ -submodule $I^{\leq d} = I \cap Z^{\leq d} \subseteq Z^{\leq d}$ is also a finitely generated right $R[T]$ -module. We also have $I = \bigcup_d I^{\leq d}$, because every element in I has the form $r_{0,0} + \cdots + r_{d,d} F^d T^d$ for some sufficiently large d , so it will lie in $Z^{\leq d}$ and I simultaneously. \square

Lemma 2.17 (Emerton's Key Lemma, right analogue). *Suppose*

- R is a Noetherian commutative \mathbf{F}_q -algebra,
- I is a right ideal in $R[F][T]$,
- R is F -finite.

Then I is a finitely generated right ideal in $R[F][T]$ if and only if the right $R[T]$ -module $\text{coker}(\tilde{I} \xrightarrow{F} I)$ is finitely generated.

Proof. (1) Suppose I is a finitely generated right ideal in $R[F][T]$. Then there exists some $n \geq 0$ such that the diagram of right $R[T]$ -modules

$$\begin{array}{ccccccc} 0 & \rightarrow & \tilde{K} & \rightarrow & \widetilde{R[F][T]^n} & \rightarrow & \tilde{I} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & K & \rightarrow & R[F][T]^n & \rightarrow & I \rightarrow 0 \end{array} \cdot F$$

commutes. Recall that the right action twist means that $\tilde{m} \cdot_{\tilde{M}} s T^j := \tilde{m} \cdot_M s^q T^j$ in the top row. The snake lemma gives us a surjection

$$(R[F][T]/R[F][T]F)^n \twoheadrightarrow \text{coker}(\tilde{I} \xrightarrow{F} I)$$

(³). There is an obvious right $R[T]$ -module isomorphism⁴

$$(2.3) \quad \begin{aligned} R[F][T]/R[F][T]F &\longrightarrow R[T] \\ \sum r_{i,\ell} F^i T^\ell &\longmapsto \sum r_{0,\ell} T^\ell. \end{aligned}$$

Note that for $s \in R$, we get

$$r_{i,\ell} F^i T^\ell \cdot s = r_{i,\ell} s^{q^i} F^i T^\ell \longmapsto r_{0,\ell} s T^\ell = r_{0,\ell} T^\ell \cdot s.$$

We conclude that the cokernel is a finitely generated right $R[T]$ -module.

(2) Suppose $\text{coker}(\tilde{I} \xrightarrow{F} I)$ is a finitely generated right $R[T]$ -module. We consider the maps of right $R[T]$ -modules

$$(2.4) \quad I^{\leq d} \longrightarrow \text{coker}(\tilde{I} \xrightarrow{F} I).$$

The images, along $d \rightarrow +\infty$, form an ascending chain of right $R[T]$ -submodules. As we had assumed that the right-hand side module was right finitely generated over $R[T]$ and thus Noetherian (since R and thus $R[T]$ are Noetherian), this chain must become stationary. On the other hand, taking the union over all d , we get the entire image of I , but I surjects onto the cokernel. Thus, we learn that there exists some d_0 such that the map in line 2.4 is surjective for all $d \geq d_0$. Next, we claim that

$$(2.5) \quad I^{\leq d} \subseteq I^{\leq d_0} + IF.$$

This is clear: By line 2.4 every element in I/IF comes from $I^{\leq d_0}$, so in I it lies in $I^{\leq d_0} + IF$. Moreover, we claim that

$$(2.6) \quad IF \cap I^{\leq d} = I^{\leq d-1} F.$$

(Proof: Right to left: Every element in $I^{\leq d-1} F$ clearly lies in IF , as well as $I^{\leq d}$. For the converse direction: Suppose $\alpha \in IF \cap I^{\leq d}$. Then we have a presentation

$$\alpha = \sum_{\ell} \sum_{i=0}^d r_{i,\ell} F^i T^\ell = \beta F \quad \text{with} \quad \beta = \sum_{\ell} \sum_{i=0}^{\infty} b_{i,\ell} F^i T^\ell \in I.$$

Thus, by the uniqueness of presentations as left polynomials in F , we get

$$\sum_{\ell} \sum_{i=0}^d r_{i,\ell} F^i T^\ell = \sum_{\ell} \sum_{i=0}^{\infty} b_{i,\ell} F^{i+1} T^\ell$$

and conclude that $b_{i,\ell} = 0$ for all $i \geq d$. So

$$\alpha = \left(\sum_{\ell} \sum_{i=0}^{d-1} b_{i,\ell} F^i T^\ell \right) F \in I^{\leq d-1} F.$$

This finishes the proof of the sub-claim.) Line 2.5 reads

$$I^{\leq d} \subseteq I^{\leq d_0} + IF.$$

(*Inductive reduction process*) For $\alpha \in I^{\leq d}$ (with $d > d_0$), we find some $\gamma \in I^{\leq d_0}$ such that

$$\alpha = (\alpha - \gamma) + \underset{\in IF}{\gamma} \quad \underset{\in I^{\leq d_0}}{\gamma}.$$

³Note that the image of $\widetilde{R[F][T]}$ under right multiplication by F is $R[F][T]F$, and not $\widetilde{R[F][T]}F$.

⁴Actually this is even a right $R[F][T]$ -module isomorphism, where F acts as $r \cdot F = 0$ for all $r \in R[T]$.

Moving γ to the other side of the equation, the left hand side lies in $I^{\leq d}$, while the right-hand side lies in IF . Thus, both lie in the intersection of both ideals. We deduce $\alpha - \gamma \in IF \cap I^{\leq d} = I^{\leq d-1}F$ (by Equation 2.6). In other words, we can promote the above equation to

$$(2.7) \quad \alpha = \underbrace{(\alpha - \gamma)}_{\in I^{\leq d-1}F} + \underbrace{\gamma}_{\in I^{\leq d_0}} = \underbrace{\hat{\alpha}}_{\in I^{\leq d-1}} \cdot F + \underbrace{\gamma}_{\in I^{\leq d_0}}$$

for a suitably chosen $\hat{\alpha} \in I^{\leq d-1}$. Now repeat this reduction process for the element $\hat{\alpha}$. This is possible unless we reach $d-1 = d_0$. In this case, the above equation literally says $\alpha \in I^{\leq d_0} \cdot R[F]$. Now, it follows that in Equation 2.7 we always get $\alpha \in I^{\leq d_0} \cdot R[F]$, for every α in this inductive reduction process.

In particular, we conclude $\alpha \in I^{\leq d_0} \cdot R[F]$ for our original α that started the reduction process. As this works for all elements $\alpha \in I^{\leq d}$, we obtain $I^{\leq d} \subseteq I^{\leq d_0} \cdot R[F]$. Taking the union over all d , we obtain $I \subseteq I^{\leq d_0} \cdot R[F]$. But we also have $I^{\leq d_0} \cdot R[F] \subseteq I$, so $I = I^{\leq d_0} \cdot R[F]$ and since $I^{\leq d_0}$ is a finitely generated right $R[T]$ -module (by Lemma 2.16), it follows that $I^{\leq d_0} = \langle a_1, \dots, a_r \rangle R[T]$ and thus $I = \langle a_1, \dots, a_r \rangle R[T] \cdot R[F]$. Since $R[T] \cdot R[F] \subseteq R[F][T]$, it follows that I is a finitely generated $R[F][T]$ -module. \square

Lemma 2.18. *Suppose*

- *R is a Noetherian commutative \mathbf{F}_q -algebra,*
- *I is a right ideal in $R[F][T]$,*
- *R is F -finite.*

A finitely generated right $R[F][T]$ -module M is finitely presented iff the right $R[T]$ -module $\ker(\widetilde{M} \xrightarrow{\cdot F} M)$ is finitely generated.

Proof. As M is a finitely generated right $R[F][T]$ -module, we get a commutative diagram

$$(2.8) \quad \begin{array}{ccccccc} 0 & \rightarrow & \widetilde{K} & \rightarrow & \widetilde{R[F][T]^n} & \rightarrow & \widetilde{M} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & K & \rightarrow & R[F][T]^n & \rightarrow & M \rightarrow 0 \end{array} \cdot F$$

and the K on the left is just defined as the kernel. Therefore, by the snake lemma

$$0 \rightarrow \ker(\widetilde{M} \xrightarrow{\cdot F} M) \rightarrow \text{coker}(\widetilde{K} \xrightarrow{\cdot F} K) \rightarrow R[T]^n \rightarrow \text{coker}(\widetilde{M} \xrightarrow{\cdot F} M) \rightarrow 0$$

is an exact sequence of right $R[F][T]$ -modules (for the middle cokernel we have used the isomorphism of line 2.3, the right action of F on $R[T]$ is tacitly understood to be the zero map). In other words, we get an exact sequence of right $R[T]$ -modules

$$0 \rightarrow \ker(\widetilde{M} \xrightarrow{\cdot F} M) \rightarrow \text{coker}(\widetilde{K} \xrightarrow{\cdot F} K) \rightarrow \ker(R[T]^n \rightarrow \text{coker}(\widetilde{M} \xrightarrow{\cdot F} M)) \rightarrow 0.$$

As $R[T]^n$ is finitely generated and $(R$ and therefore) $R[T]$ is Noetherian, it follows that the kernel on the right, is also right finitely generated over $R[T]$. It follows that $\ker(\widetilde{M} \xrightarrow{\cdot F} M)$ is finitely generated over $R[T]$ if and only if $\text{coker}(\widetilde{K} \xrightarrow{\cdot F} K)$ is finitely generated over $R[T]$. However, K is a right $R[F][T]$ -submodule of $R[F][T]^n$, so by Lemma 2.17 it is a finitely generated right $R[F][T]$ -module iff $\text{coker}(\widetilde{K} \xrightarrow{\cdot F} K)$ is a finitely generated right $R[T]$ -module (the Lemma is about right ideals, but this is easily seen to be equivalent to hold for all right submodules of free right modules). However, by Diagram 2.8, K being right finitely generated over $R[F][T]$, is equivalent to M being finitely presented as a right $R[F][T]$ -module. \square

The following is the analogue of Emerton's theorem, Theorem 2.7, just right instead of left, and stabilized along additional polynomial variables.

Proposition 2.19. *Suppose*

- *R is a Noetherian commutative \mathbf{F}_q -algebra, and*
- *R is F -finite.*

Then the ring $R[F]$ is right stably coherent.

Proof. (Step 1) Firstly, we just prove that $R[F][T]$ is right coherent. Let I be a finitely generated right ideal in $R[F][T]$. Then by the snake lemma

$$I \hookrightarrow R[F][T]$$

induces $\ker(\tilde{I} \xrightarrow{F} I) \hookrightarrow \ker(\widetilde{R[F][T]} \xrightarrow{F} R[F][T]) = 0$, so the kernel is zero and thus certainly right finitely generated over $R[F][T]$. By Lemma 2.18 it follows that I is finitely presented. Thus, every finitely generated right ideal is even finitely presented, i.e. $R[F][T]$ is right coherent.

(Step 2) We prove right stable coherence by reducing to finitely many variables (this is inspired by the proof of [Ger74, Thm. 1.8], who does the same for non-commuting variables): Following Remark 2.9, Step 1 actually shows that $R[F][T_i]_{i \in I}$ is right coherent for any *finite* set I . Now, for an arbitrary set I we have the filtering colimit presentation

$$R[F][T_i]_{i \in I} = \varinjlim_{I' \subseteq I, I' \text{ finite}} R[F][T_i]_{i \in I'}.$$

For an arbitrary ring A , every non-zero free right A -module is faithfully flat over A . Since the polynomial rings $A[T_i]_{i \in I}$ are right flat over A (unlike, in general, their Frobenius counterpart $A[F]$), it follows that the above colimit has all transition morphisms (faithfully) flat. By [Sou70, §5, Prop. 20] a flat colimit of right coherent rings is again right coherent. This shows that $R[F][T_i]_{i \in I}$ is right coherent, regardless the choice of I . \square

2.3. Right regularity. The following result is due to Linquan Ma [Ma14, Theorem 3.2]:

Proposition 2.20 (Ma). *Suppose that R is a commutative \mathbf{F}_q -algebra of finite global dimension. Then*

$$\text{gldim}_{\text{right}} R[F][T_1, \dots, T_s] \leq \text{gldim}_{\text{right}} R + s + 1.$$

Ma's paper [Ma14] additionally assumes that R is F -finite and obtains an equality. However, for our purposes the stated estimate suffices. As we want to cite an ingredient of the proof in the K -theory computations later on, we give his proof adapted to our own notation:

Let M be a right R -module. Then we can define a right $R[F]$ -module

$$(2.9) \quad M[X] := M \otimes_R R[F].$$

This can be spelled out as follows:

$$\begin{aligned} M[X] &= \left\{ \sum m_i X^i \mid m_i \in M, m_i = 0 \text{ for all but finitely many } i \right\} \\ (m_i X^i) \cdot r F^j &:= m_i r^{q^i} X^{i+j} = (m_i \cdot_M r^{q^i}) X^{i+j}. \end{aligned}$$

Now, suppose N is an arbitrary right $R[F]$ -module. Then, we can also interpret it as a right R -module so that $N[X]$ makes sense. We define $\phi : N[X] \rightarrow N$, $nX^i \mapsto nF^i$. This

is a right $R[F]$ -module homomorphism. Since $nX^0 \mapsto n$, for all $n \in N$, it is surjective. Next, for every right $R[F]$ -module N , we define a new right $R[F]$ -module \tilde{N} . Its additive group is just the one of N , but we change the right action to $(\tilde{n}) \cdot_{\tilde{N}} rF^j := \tilde{n} \cdot_N r^q F^j$, where we refer to the original right $R[F]$ -module structure of N on the right hand side. One checks that this is indeed a right $R[F]$ -module structure. Next, we define a map $\psi : \tilde{N}[X] \rightarrow N[X]$, $nX^i \mapsto nX^{i+1} - nFX^i$. We claim that this is a morphism of right $R[F]$ -modules. This is a little delicate because of the different module structures. We compute

$$(nX^i) \cdot_{\tilde{N}[X]} rF^j = (n \cdot_{\tilde{N}} r^{q^i})X^{i+j} = (n \cdot_N r^{q^{i+1}})X^{i+j},$$

where on the right-hand side we just have the ordinary right $R[F]$ -module structure, so we could also just write $nr^{q^{i+1}}X^{i+j}$. Now, $\psi(nr^{q^{i+1}}X^{i+j}) = nr^{q^{i+1}}X^{i+j+1} - nr^{q^{i+1}}FX^{i+j}$. On the other hand, $\psi(nX^i) \cdot_{N[X]} rF^j = (nX^{i+1} - nFX^i) \cdot rF^j = nr^{q^{i+1}}X^{i+j+1} - nr^{q^{i+1}}FX^{i+j}$. Thus, it is indeed a right $R[F]$ -module homomorphism. This yields a ‘Frobenius-twisted Koszul complex’:

Lemma 2.21. *For every right $R[F]$ -module N ,*

$$(2.10) \quad 0 \longrightarrow \tilde{N}[X] \xrightarrow{\psi} N[X] \xrightarrow{\phi} N \longrightarrow 0$$

is a short exact sequence of right $R[F]$ -modules.

Proof. We have $\phi\psi(\tilde{n}X^i) = \phi(nX^{i+1} - nFX^i) = nF^{i+1} - nF^{i+1} = 0$. We need to study the kernel of ψ . Suppose $\alpha = \sum_{i=0}^m n_i X^i$ lies in the kernel.

$$\psi\left(\sum_{i=0}^m n_i X^i\right) = \sum_{i=0}^m n_i X^{i+1} - \sum_{i=0}^m n_i F X^i$$

and since the largest X -degree on the right-hand side, namely X^{m+1} , has coefficient n_m , we must have $n_m = 0$ if this element indeed lies in the kernel of ψ . Thus, we could have started this computation with $m-1$ instead of m . By induction, it follows that $\alpha = 0$. Finally, we need to show exactness in the middle, i.e. that every element in the kernel of ϕ lies in the image of ψ . Suppose $\alpha = \sum_{i=0}^m n_i X^i$ lies in $\ker \phi$ and $m \geq 2$. Then

$$\alpha - \psi(n_m X^{m-1}) = \sum_{i=0}^m n_i X^i - (n_m X^m - n_m F X^{m-1})$$

is has X -degree strictly less than α , and ϕ sends this new element still to zero (since by assumption it sends α to zero, and we already know that $\phi\psi$ is the zero map). Thus, by induction, we can split off pre-images from α unless $m = 1$. So let us restrict to this case and say $\alpha = n_1 X + n_0$. We find $\phi(\alpha) = n_1 F + n_0$, so if this is zero, we have $n_0 = -n_1 F$, i.e. $\alpha = n_1 X - n_1 F$. Now, we obviously have $\psi(n_1) = \alpha$, so this α also has a pre-image. \square

Proof of Prop. 2.20. Suppose $s = 0$. By Equation 2.10 there is a quasi-isomorphism from the 2-term complex $\left[\tilde{N}[X] \rightarrow N[X]\right]_{-1,0}$ to N . Thus, our claim is proven once we can show that each right $R[F]$ -module of the shape $N[X]$ has a projective resolution of length at most n . By taking the cone, this then yields a projective resolution of N of length at most $n+1$. Suppose N is a finitely generated right R -module. Note that if

$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence, so is $0 \rightarrow A[X] \rightarrow B[X] \rightarrow C[X] \rightarrow 0$ (as $R[F]$ is flat as a left R -module, cf. Equation 2.9). Moreover, there is an isomorphism of right $R[F]$ -modules $R[X] \rightarrow R[F]$, $rX^i \mapsto rF^i$, showing that $R[X]$ is a rank one free $R[F]$ -module, and thus if R is a free R -module, $R[X]$ is a free R -module. If N is a projective R -module, $N \oplus M$ is a free R -module for a suitable M . Thus, $N[X] \oplus M[X]$ is free. It follows that $N[X]$ is a projective right $R[F]$ -module. As a result of preserving exact sequences, this means that every projective resolution of length n of a right R -module N induces a projective resolution of length n of the right $R[F]$ -module $N[X]$. Finally, we use the Hilbert syzygy theorem ([Lam99, Ch. 2, §5, (5.36), Theorem]) to deduce

$$\text{gldim}_{\text{right}} R[F][T_1, \dots, T_s] = \text{gldim}_{\text{right}} R[F] + s,$$

proving the claim. \square

Lemma 2.22. *Suppose R is a commutative F -finite Noetherian \mathbf{F}_q -algebra. If N is a right $R[F]$ -module, which is finitely generated as a right R -module, then $N[X]$, $\tilde{N}[X]$ and N are finitely presented right $R[F]$ -modules.*

Proof. As N is finitely generated and R Noetherian, N is even finitely presented, say $0 \rightarrow K \rightarrow R^{\oplus m} \rightarrow N \rightarrow 0$ is exact with K finitely generated. Hence, $0 \rightarrow K[X] \rightarrow R[X]^{\oplus m} \rightarrow N[X] \rightarrow 0$ is also exact. Since $R[X] \cong R[F]$ and because a finite set of generators of K as an R -module will induce a finite set of generators of $K[X]$ as a right $R[F]$ -module, it follows that $N[X]$ is a quotient of a free finite rank $R[F]$ -module by a finitely generated one, and thus $N[X]$ is finitely presented. By Lemma 2.13 (or Corollary 2.14) it follows that $\tilde{N}[X]$ is also finitely presented. Sequence 2.10 now implies that N is the quotient of two finitely presented modules, and thus itself finitely presented. \square

3. SHEAF-THEORETIC PROPERTIES OF THE FROBENIUS SKEW RING

3.1. Construction. It is very important that the Frobenius skew ring $R[F]$ can be turned into a Zariski sheaf. This is well-known, but since it plays an important rôle for us, let us recall some details. Again, Yoshino's paper [Yos94] is an excellent reference. Firstly, if S is a multiplicative set in R , it is *not* central in $R[F]$. Thus, it is a priori not even clear whether a localization at S exists. One needs to check the Ore conditions [Coh91, Ch. 9, §9.1], [Coh95, §1.3], [Lam99, §10A].

Lemma 3.1. *Every multiplicative subset $S \subseteq R$ satisfies the left- and right denominator set axioms as a multiplicative subset of $R[F]$.*

Proof. This is well-known. As it is absolutely crucial for the following, we prove the two key axioms: (*S is left permutable*) For every $s \in S$ and $r \in R$ there exists some $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{r}s = \tilde{s}r$. This can be checked as follows: Given $r = \sum_{i=0}^n r_i F^i$ with $r_i \in R$ define $\rho_i := r_i s^{q^n - q^i}$ and $\tilde{r} := \sum_{i=0}^n \rho_i F^i$ as well as $\tilde{s} := s^{q^n}$. Then

$$\tilde{r}s = \sum \rho_i F^i s = \sum r_i s^{q^n - q^i} s^{q^i} F^i = \sum r_i s^{q^n} F^i = s^{q^n} \sum r_i F^i = \tilde{s}r$$

as desired.⁵ (*S is right permutable*) For every $s \in S$ and $r \in R$ there exists some $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $r\tilde{s} = \tilde{r}s$. Using the notation as before, define $\tilde{s} := s$ and

⁵Note that this proof only works because there exists some finite degree n . It would fail for Frobenius skew power series $R[[F]]$.

$\rho_i := r_i s^{q^i - 1}$. For $i = 0$, the latter means $\rho_0 = r_0$. Then

$$s\tilde{r} = \sum s\rho_i F^i = \sum r_i s^{q^i} F^i = \sum r_i F^i s = r\tilde{s}.$$

One also needs to check whether S is left or right reversible, and we leave the very similar verification to the reader (if R is a domain, these conditions are empty). \square

These conditions being checked, it follows that the left localization $S^{-1}R[F]$ and right localization $R[F]S^{-1}$ both exist. In fact, they agree:

Lemma 3.2 ([Yos94, (4.10), Prop.]). *Let R be a commutative ring. Suppose S is a multiplicative subset. Then there are isomorphisms*

$$R[F]S^{-1} \cong S^{-1}R[F] \cong (S^{-1}R)[F] \cong S^{-1}R \otimes_R R[F] \cong R[F] \otimes_R S^{-1}R.$$

This lemma implies that $\mathcal{O}_X[F]$ can be turned into a quasi-coherent sheaf of \mathcal{O}_X -modules; and for this it does not matter whether we let \mathcal{O}_X act from the left or the right (even though these two actions are different!). It becomes a sheaf of \mathcal{O}_X -bimodules over \mathbf{F}_q , or equivalently a sheaf of left $(\mathcal{O}_X \otimes_{\mathbf{F}_q} \mathcal{O}_X)$ -modules with

$$(\alpha, \beta) \cdot m := \alpha m \beta.$$

Note that it is *not* a sheaf of \mathcal{O}_X -algebras since the natural inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X[F]$ as constant polynomials does not lie in the center of the ring.

Corollary 3.3. *$\mathcal{O}_X[F]$ is a quasi-coherent sheaf of \mathcal{O}_X -modules, and this structure is indifferent to whether we let \mathcal{O}_X act from the left or from the right. It is also a sheaf of \mathbf{F}_q -algebras, but (usually) not of \mathcal{O}_X -algebras.*

Remark 3.4. In [EK04b] the sheaf $\mathcal{O}_X[F]$ is usually denoted by $\mathcal{O}_{F,X}$.

3.2. Categories of sheaves. Next, we shall introduce some categories mimicking ordinary coherent \mathcal{O}_X -modules, but with an additional right action by the Frobenius (so the action looks like a Cartier operator). Because the finite presentation conditions will be with respect to $\mathcal{O}_X[F]$ instead of \mathcal{O}_X , these can be much bigger than the coherent Cartier modules of [BB11].

Definition 3.5. *Let X/\mathbf{F}_q be an F -finite Noetherian separated scheme.*

- (1) *Denote by $\mathbf{Coh}_{\mathcal{O}_X[F]}(X)$ the category whose objects are locally finitely presented right $\mathcal{O}_X[F]$ -module sheaves, whose underlying (right) \mathcal{O}_X -module sheaves are quasi-coherent. Morphisms are arbitrary right $\mathcal{O}_X[F]$ -module morphisms.*
- (2) *If $Z \subseteq X$ is a closed subset, define the full subcategory $\mathbf{Coh}_{\mathcal{O}_X[F], Z}(X)$ which consists of those right $\mathcal{O}_X[F]$ -module sheaves whose support, in terms of the underlying right \mathcal{O}_X -module sheaf, is contained in Z .*

As the rings $R[F]$ are right coherent, we will of course expect to get an abelian category this way:

Lemma 3.6. *Let X/\mathbf{F}_q be an F -finite Noetherian separated scheme. Then*

- (1) *the category $\mathbf{Coh}_{\mathcal{O}_X[F]}(X)$ is an abelian category, and*
- (2) *for every closed subset $Z \subseteq X$, the category $\mathbf{Coh}_{\mathcal{O}_X[F], Z}(X)$ is a Serre subcategory. In particular, $\mathbf{Coh}_{\mathcal{O}_X[F], Z}(X)$ is itself an abelian category.*

Proof. (1) Affine locally, the ring $\mathcal{O}_X[F](U) = R[F]$ for $R := \mathcal{O}_X(U)$ is right coherent by Prop. 2.19 and X being F -finite. Affine locally, right $\mathcal{O}_X[F]$ -module sheaves then correspond to a finitely presented right $R[F]$ -modules. By Prop. 2.3 the latter category is abelian. Globally, the category of all right $\mathcal{O}_X[F]$ -module sheaves (with no more conditions) is Grothendieck abelian. One shows that $\mathbf{Coh}_{\mathcal{O}_X[F]}(X)$ is closed under kernels and cokernels in it. (2) For any quasi-coherent sheaf of \mathcal{O}_X -modules \mathcal{F} the *support* is defined as the set of scheme points $x \in X$ such that $\mathcal{F}_x \neq 0$ (this definition is more customary in the context of coherent sheaves, where \mathcal{F} is additionally known to be a closed subset. For quasi-coherent sheaves the support can be arbitrary). It is easy to see that for a short exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ of such sheaves, one has $\text{supp } \mathcal{F} = \text{supp } \mathcal{F}' \cup \text{supp } \mathcal{F}''$. This immediately implies that $\mathbf{Coh}_{\mathcal{O}_X[F], Z}(X)$ is a Serre subcategory. \square

As in the classical case, the K -theory for these Frobenius skew-rings does not see nil-thickenings:

Lemma 3.7 (Dévissage for $\mathcal{O}_X[F]$). *Let X/\mathbf{F}_q be an F -finite Noetherian separated scheme, $i : Z \hookrightarrow X$ a closed subscheme. Then the inclusion of categories*

$$\mathbf{Coh}_{\mathcal{O}_Z[F]}(Z) \hookrightarrow \mathbf{Coh}_{\mathcal{O}_X[F], Z}(X)$$

via pushforward, induces an equivalence in K -theory.

$$K(\mathbf{Coh}_{\mathcal{O}_Z[F]}(Z)) \xrightarrow{\sim} K(\mathbf{Coh}_{\mathcal{O}_X[F], Z}(X)).$$

Proof. We adapt the proof of the analogous result for \mathcal{O}_X -module sheaves. We begin with the following observation: Let R be any commutative \mathbf{F}_q -algebra and $I \subseteq R$ an ideal. Then the set

$$I \cdot R[F] = \left\{ \sum r_i F^i \mid r_i \in I \right\}$$

is a two-sided ideal in $R[F]$ (since $s \cdot r_i F^i = (sr_i) F^i$ and $r_i F^i \cdot s = (r_i s^{q^i}) F^i$ for all $s \in R$). We get

$$\iota^\# : R[F] \longrightarrow R[F] / I \cdot R[F] \xrightarrow{\sim} (R/I)[F],$$

which generalizes to the respective sheaves of algebras $\mathcal{O}_X[F]$ and $\mathcal{O}_Z[F]$, i.e. we may read $\iota^\#$ as corresponding to the closed immersion i affine locally. Moreover, affine locally, the pushforward interprets a right $(R/I)[F]$ -module as a right $R[F]$ -module via $\iota^\#$. This defines an exact functor. Let \mathcal{I}_Z denote the ideal sheaf defining the closed immersion i . As the support for a sheaf in $\mathbf{Coh}_{\mathcal{O}_X[F], Z}(X)$ was defined on the level of the underlying quasi-coherent \mathcal{O}_X -module sheaf, every such sheaf can be presented as an $\mathcal{O}_X[F]/\mathcal{I}_Z^m[F] \cong (\mathcal{O}_X/\mathcal{I}_Z^m)[F]$ -module sheaf for some sufficiently large $m \geq 1$. In particular, we can filter the sheaf according to the powers of $\mathcal{I}_Z^m[F]$. Thus, every object in $\mathbf{Coh}_{\mathcal{O}_X[F], Z}(X)$ admits a finite filtration whose graded pieces are annihilated by $\mathcal{I}_Z^m[F]$, and thus lie in the full sub-category of $\mathcal{I}_Z[F]$ -annihilated sheaves. This category is non-empty, full, closed under subobjects, quotients and finite direct sums. Thus, the assumptions of Quillen's dévissage theorem are satisfied, [Qui73, §5, Theorem 4]. However, the latter is also equivalent to the category of finitely presented right $(\mathcal{O}_X/\mathcal{I}_Z)[F]$ -module sheaves whose underlying \mathcal{O}_X -module sheaves are quasi-coherent, i.e. $\mathbf{Coh}_{\mathcal{O}_Z[F]}(Z)$, giving the claim. \square

Remark 3.8. It is important to work with $I \cdot R[F]$. Choosing the reverse order, $R[F] \cdot I$, virtually never yields a two-sided ideal, only a left ideal.

In order to proceed, it will be necessary to relate the quotient category

$$\mathrm{Coh}_{\mathcal{O}_X[F]}(X)/\mathrm{Coh}_{\mathcal{O}_X[F],Z}(X)$$

to the open complement $U := X - Z$. Recall the following fact due to Gabriel, which explains the interplay of these categories in the classical case:

Proposition 3.9 ([Gab62, §III.5, Prop. VI.3]). *Suppose X is a scheme, $U \subseteq X$ an open subset such that the open immersion $j : U \hookrightarrow X$ is a quasi-compact morphism.*

- (1) *Then for any quasi-coherent \mathcal{O}_U -module sheaf \mathcal{F} on U , the pushforward $j_*\mathcal{F}$ is a quasi-coherent \mathcal{O}_X -module sheaf, and this functor is right adjoint to the pullback j^* so that*

$$j^* \dashv j_*$$

is a pair of adjoint functors. Moreover, j_ is fully faithful.*

- (2) *Let $\ker j^*$ be the full subcategory of objects $\mathcal{F} \in \mathrm{QCoh}(X)$ such that $j^*\mathcal{F}$ is a zero object. Then $\ker j^*$ is precisely the Serre subcategory of quasi-coherent \mathcal{O}_X -module sheaves with support contained in the closed set $Z := X - U$, i.e.*

$$\ker j^* = \mathrm{QCoh}_Z(X).$$

- (3) *There is an equivalence of abelian categories*

$$\mathrm{QCoh}(U) \xrightarrow{\sim} \mathrm{QCoh}(X)/\mathrm{QCoh}_Z(X).$$

Proposition 3.10 (Localization for $\mathcal{O}_X[F]$). *Let X/\mathbf{F}_q be an F -finite Noetherian separated scheme, $i : Z \hookrightarrow X$ a reduced closed subscheme and $U := X - Z$ the open complement. Write $j : U \hookrightarrow X$ for the open immersion. Then there is a homotopy fiber sequence*

$$(3.1) \quad K(\mathrm{Coh}_{\mathcal{O}_Z[F]}(Z)) \xrightarrow{i_*} K(\mathrm{Coh}_{\mathcal{O}_X[F]}(X)) \xrightarrow{j^*} K(\mathrm{Coh}_{\mathcal{O}_U[F]}(U)) \longrightarrow +1.$$

Proof. This is of course just the Frobenius skew ring analogue of Quillen's localization theorem [Qui73, Theorem 5]. In Quillen's setup, he uses that the coherent sheaves of \mathcal{O}_X -modules, $\mathrm{Coh}_{\mathcal{O}_X,Z}(X)$, form a Serre subcategory of the abelian category $\mathrm{Coh}_{\mathcal{O}_X}(X)$. By Gabriel's result, Prop. 3.9, the pullback along j induces an equivalence of abelian categories

$$j^* : \mathrm{Coh}_{\mathcal{O}_X}(X)/\mathrm{Coh}_{\mathcal{O}_X,Z}(X) \xrightarrow{\sim} \mathrm{Coh}_U(U).$$

The functor j^* is exact since it is a localization and thus $(j^*\mathcal{O}_X)(V)$ is always a flat $\mathcal{O}_X(V)$ -module for every open $V \subseteq X$. Thus, in order to prove our claim above, we just need to adapt Quillen's proof for the modules over the sheaf $\mathcal{O}_X[F]$ instead of \mathcal{O}_X . Let us follow this path step by step:

(Step 1) By Lemma 3.6 we again have a Serre subcategory of an abelian category, so Quillen's localization theorem [Qui73, Theorem 5] for Serre subcategories in abelian categories readily produces a homotopy fiber sequence

$$(3.2) \quad K(\mathrm{Coh}_{\mathcal{O}_X[F],Z}(X)) \xrightarrow{i_*} K(\mathrm{Coh}_{\mathcal{O}_X[F]}(X)) \xrightarrow{q} K(\mathrm{Qu}) \longrightarrow +1,$$

where $\mathrm{Qu} := \mathrm{Coh}_{\mathcal{O}_X[F]}(X)/\mathrm{Coh}_{\mathcal{O}_X[F],Z}(X)$ denotes the quotient abelian category and q the canonical exact functor to the quotient category (see [Gab62, Ch. III, §1, Prop. 1] for the latter). In our situation, the functor

$$(3.3) \quad j^* : \mathrm{Coh}_{\mathcal{O}_X[F]}(X) \longrightarrow \mathrm{Coh}_{\mathcal{O}_U[F]}(U)$$

is also exact because $(j^*\mathcal{O}_X[F])(V)$ is also always a flat $\mathcal{O}_X[F](V)$ -module thanks to Lemma 3.2. We claim that this functor descends to an exact functor

$$(3.4) \quad j^* : \mathbf{Qu} \longrightarrow \mathbf{Coh}_{\mathcal{O}_U[F]}(U).$$

This is clear: We need to look at the underlying quasi-coherent \mathcal{O}_X -module sheaves. Quasi-coherent sheaves with support in Z clearly go to a zero object on U . Moreover, if a sheaf $\mathcal{F} \in \mathbf{Qu}$ lies in the kernel of j^* , it also lies in the kernel of j^* on the level of the underlying quasi-coherent \mathcal{O}_X -module sheaves, so again by Prop. 3.9, \mathcal{F} must have support contained in Z . Hence, we obtain that j^* in line 3.3 factors over the quotient \mathbf{Qu} , giving line 3.4 (see [Gab62, Ch. III, §1, Cor. 2] for this factorization result). The functor j^* is also fully faithful since the latter reduces to the same property for the underlying quasi-coherent \mathcal{O}_X -module sheaves.

(Step 2) Next, we claim that $j^* : \mathbf{Qu} \rightarrow \mathbf{Coh}_{\mathcal{O}_U[F]}(U)$ is essentially surjective. For this, one can copy and adapt the standard proof that (on any quasi-separated and quasi-compact scheme) every finitely presented \mathcal{O}_X -module sheaf on U has a finitely presented extension to X , being a pre-image under j^* , [Sta16, Tag 01PD]. Let $\mathcal{F} \in \mathbf{Coh}_{\mathcal{O}_U[F]}(U)$ be given. As our X is even Noetherian, we can pick a finite Zariski open cover $X = \bigcup_{i=1}^{\ell} V_i$. One can write $X = U \cup \bigcup_{i=1}^{\ell} V_i$ and thus if one can successively extend \mathcal{F} from U to $U \cup V_1$, and then to $U \cup V_1 \cup V_2$ etc., our claim is proven. Thus, it suffices to deal with the case such that $X = U \cup V$ with V affine. For this problem, it suffices to extend \mathcal{F} from $U \cap V$ to V , since on $U \setminus (U \cap V)$ the sheaf \mathcal{F} is known and there are no glueing conditions that have to be met. Moreover, we can reduce to the case where U is affine, because we may cover U by such opens and if we solve the extension problem on these, we are done. Now, as we have reduced the problem to an affine situation, we may cover $U \cap V$ inside V by a finite number of distinguished affine opens, say $U \cap V = \bigcup_{j=1}^k D(f_j)$, where f_j are the functions whose non-vanishing loci are the $D(f_j)$. As \mathcal{F} is known to be a finitely generated right $\mathcal{O}_U[F]$ -module by assumption, this remains true on the opens $\iota_j : D(f_j) \hookrightarrow U$. Now, by Lemma 3.2 we have

$$\begin{aligned} \mathcal{F}(D(f_j)) &= \mathcal{F}(U) \otimes_{\mathcal{O}_U[F]} \mathcal{O}_{D(f_j)}[F] \\ &\cong \mathcal{F}(U) \otimes_{\mathcal{O}_U[F]} \mathcal{O}_U[F] \otimes_{\mathcal{O}_U} \mathcal{O}_U\left[\frac{1}{f_j}\right] \cong \mathcal{F}(U) \otimes_{\mathcal{O}_U} \mathcal{O}_U\left[\frac{1}{f_j}\right]. \end{aligned}$$

Thus, a finite set of generators will affine locally have the shape $a_i \otimes \frac{1}{f_j^i}$ ($a_i \in \mathcal{F}(U)$), for a finite index set $i \in I$, and so if we take the sub-module generated by the a_i inside $\mathcal{F}(U)$, we get a finitely generated right $\mathcal{O}_U[F]$ -module $\mathcal{G}_j \subseteq \mathcal{F}$ whose pullback to $D(f_j)$ satisfies $\iota_j^* \mathcal{G}_j \xrightarrow{\sim} \iota_j^* \mathcal{F}$. In a similar way, one can lift not just generators, but also a presentation, cf. [Sta16, Tag 01PD].

(Step 3) Combining the previous steps, we learn that the functor in line 3.4 is exact, fully faithful and essentially surjective. Thus, it is an equivalence of categories. Thus, the homotopy fiber sequence in line 3.2 transforms into

$$K(\mathbf{Coh}_{\mathcal{O}_X[F], Z}(X)) \xrightarrow{i_*} K(\mathbf{Coh}_{\mathcal{O}_X[F]}(X)) \xrightarrow{j^*} K(\mathbf{Coh}_{\mathcal{O}_U[F]}(U)) \longrightarrow +1.$$

By our version of dévissage, Lemma 3.7, we arrive at our claim in line 3.1. This finishes the proof. \square

The following is of course also entirely analogous to the corresponding statement for the K -theory of coherent \mathcal{O}_X -module sheaves.

Corollary 3.11. *Let X/\mathbf{F}_q be an F -finite Noetherian separated scheme. The presheaf of spectra (see [Jar04] for background),*

$$KF(U) := K(\mathrm{Coh}_{\mathcal{O}_U[F]}(U)),$$

satisfies Zariski descent, and (equivalently) satisfies the Mayer–Vietoris property for Zariski squares.

Proof. Satisfying Zariski descent is equivalent to the Mayer–Vietoris property [Wei13, Ch. V, Theorem 10.2] (the original results for this are due to Brown and Gersten [BG73], based on Thomason’s ideas). The latter can be shown as in the case of ordinary K -theory and the presheaf of spectra $U \mapsto K(\mathrm{Coh}_{\mathcal{O}_X}(U))$. For this, one just needs to have a localization sequence, cf. [Wei13, Ch. V, Example 10.3]: For $X = U \cup V$ with $U, V \subseteq X$ open, define $Z := X - U$ and one has the closed-open complements $Z \hookrightarrow X \hookleftarrow U$ resp. $Z \hookrightarrow V \hookleftarrow U \cap V$. Using the localization sequence of Prop. 3.10 for either, one obtains the two homotopy fiber sequences

$$\begin{array}{ccccccc} KF(Z) & \longrightarrow & KF(X) & \longrightarrow & K(U) & \longrightarrow & +1 \\ & & \downarrow & & \downarrow & & \\ KF(Z) & \longrightarrow & KF(V) & \longrightarrow & KF(U \cap V) & \longrightarrow & +1 \end{array}$$

and the downward arrows are functorially induced from the pullback along the open immersion $V \hookrightarrow X$. As the left downward arrow is an equivalence, it follows that the square is homotopy bi-Cartesian. This is the required Mayer–Vietoris property. This proof is an exact copy of the classical one, just using the Frobenius variant of K -theory and the respective localization sequence. \square

3.2.1. Variation: Frobenius vector bundles. K -theory is usually defined on the basis of vector bundles or perfect complexes. A similar treatment is also conceivable in the present situation.

Lemma 3.12. *Let X/\mathbf{F}_q be an F -finite Noetherian regular separated scheme. Then*

$$K(\mathcal{O}_X[F]\mathbf{P}(X)) \cong K(\mathbf{P}_{\mathcal{O}_X[F]}(X)) \cong K(\mathrm{Coh}_{\mathcal{O}_X[F]}(X)),$$

where $\mathbf{P}_{\mathcal{O}_X[F]}$ (resp. $\mathcal{O}_X[F]\mathbf{P}$) denotes the exact categories of finitely generated projective right (resp. left) $\mathcal{O}_X[F]$ -module sheaves.

Proof. The right regularity of the ring $\mathcal{O}_X[F](X)$, implied by the uniform upper bound on projective dimension of Prop. 2.20, and its right coherence, Prop. 2.19, implies the right-hand side equivalence. Next, we recall that for any arbitrary ring A , every finitely generated (right) A -module is reflexive, i.e. the duality functor

$$(-)^\vee : \mathbf{P}_{\mathcal{O}_X[F]}(X) \rightarrow \mathcal{O}_X[F]\mathbf{P}(X), \quad M \mapsto M^\vee := \mathrm{Hom}_A(M, A)$$

is an equivalence because $\mathcal{O}_X[F]\mathbf{P}(X)$ has a corresponding duality functor, and double dualization either way tautologically is an equivalence, thanks to the reflexivity of all objects in the category. This induces an anti-equivalence between the left and right module categories, and this induces an equivalence of their K -theories. \square

Remark 3.13. I do not see any reason why one should expect that the K -theory of coherent *left* $\mathcal{O}_X[F]$ -modules ${}_{\mathcal{O}_X[F]}\mathrm{Coh}(X)$ agrees with the above K -theories as well, apart from aesthetics perhaps.

It should also be possible to prove a localization sequence in this setup, even for F -finite schemes which are not regular. The papers of Grayson [Gra80] and Weibel–Yao [WY92] discuss a suitable non-commutative localization theorem *for rings*, i.e. in our context this only helps us for the affine case. However, their constructions are entirely functorial in the ring and the multiplicative subset. With a little work, one can sheafify these localization results.

As for \mathcal{O}_X -modules, there is no counterpart of dévissage if one doesn't have a comparison to coherent sheaves available. We will not pursue this further.

3.3. Frobenius line invariance of K -theory. A fundamental fact about algebraic K -theory is its \mathbf{A}^1 -invariance, namely:

Theorem 3.14 (Bass–Quillen). *Suppose X is a Noetherian regular separated scheme. Then the pullback along the projection $\pi : X \times \mathbf{A}^1 \rightarrow X$ induces an equivalence*

$$K(X) \xrightarrow{\sim} K(X \times \mathbf{A}^1).$$

For the K -theory of coherent sheaves this remains true without the assumption that X be regular.

This is [Qui73, Theorem 8 and Corollary]. For K_0, K_1 it is due to Bass. Actually, this can be generalized to arbitrary affine fibrations π , e.g. vector bundles. However, for us the above formulation is the relevant one.

We shall prove:

Theorem 3.15. *Let X/\mathbf{F}_q be a Noetherian separated F -finite regular scheme. Then there is an equivalence in K -theory*

$$\Psi : K(\mathrm{Coh}(X)) \xrightarrow{\sim} K(\mathrm{Coh}_{\mathcal{O}_X[F]}(X)),$$

induced by right tensoring $\mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X[F]$.

Philosophy 3.16. Recall that over the base field \mathbf{F}_p of characteristic p , the Frobenius skew ring $\mathcal{O}_X[F]$ is defined by the relation $r^p F = F r$ for all sections r . If we were so bold to view a ‘field with one element’ as part of this family of fields, this relation would become $r F = F r$, i.e. F would just be an ordinary commuting variable and thus $\mathcal{O}_X[T]$ just the ordinary polynomial ring. The above Frobenius line invariance becomes ordinary \mathbf{A}^1 -invariance. This aspect is too magical to remain unsaid [Sou04].

For perfect reduced rings, where the Frobenius is an automorphism, Grayson uses a Frobenius-twisted projective line in his article [Gra88], which serves an analogous purpose.

To prove this, we use a particularly general version of Quillen’s Theorem, Theorem 1.1, which is due to Gersten [Ger74]. Gersten managed to remove the Noetherian hypothesis and replace it by a condition on the coherence of the polynomial ring:

Theorem 3.17 ([Ger74, §3, Theorem 3.1]). *Let A be an increasingly filtered ring*

$$A = \bigcup_{s \geq 0} A^{\leq s}$$

such that $A^{\leq s} \cdot A^{\leq t} \subseteq A^{\leq s+t}$. Suppose that the polynomial ring $A[T]$ is right coherent and right regular. Then the ring homomorphism $A^{\leq 0} \hookrightarrow A$ induces an equivalence in K -theory

$$K(\mathbf{P}_f(A^{\leq 0})) \xrightarrow{\sim} K(\mathbf{P}_f(A)).$$

Moreover:

Theorem 3.18 ([Ger74, §2, Theorem 2.3]). *Suppose A is a right coherent and right regular ring. Then the exact functor*

$$\mathbf{P}_f(A) \longrightarrow \mathbf{Mod}_{fp}(A)$$

induces an equivalence in K -theory.

To make sense of the functor, note that the right coherence of A implies that every finitely generated projective right module is actually coherent (Prop. 2.4). We refer to Gersten's paper for the proofs.

Proof of Theorem 3.15. (Step 1) We first deal with the special case that $X = \text{Spec } R$ is affine. Then we have equivalences of abelian categories

$$(3.5) \quad K(\text{Coh}(X)) \xrightarrow{\sim} K(\text{Mod}_{fp}(R)), \quad K(\text{Coh}_{\mathcal{O}_X[F]}(X)) \xrightarrow{\sim} K(\text{Mod}_{fp}(R[F])).$$

On the left-hand side, it does not matter whether we deal with finitely generated or finitely presented modules since R is Noetherian by assumption. Clearly the skew Frobenius ring is filtered

$$R[F]^{\leq d} = \left\{ \sum_{i=0}^d r_i F^i \mid r_i \in R \right\} \quad \text{with} \quad R[F]^{\leq 0} \cong R.$$

By Prop. 2.19 the polynomial ring $R[F][T]$ is right coherent (this is the crucial bit why just proving right coherence for $R[F]$ would not have sufficed!) and by Prop. 2.20 the regularity of R (which since R is Noetherian commutative by Serre implies that its global dimension agrees with its Krull dimension and is finite, [Lam99, (5.94) Theorem]) implies the finiteness of the right global dimension of $R[F][T]$ and thus implies right regularity. Thus, the conditions of Gersten's theorem (Theorem 3.17) are met, i.e. we get an equivalence in K -theory

$$K(\mathbf{P}_f(R)) \xrightarrow{\sim} K(\mathbf{P}_f(R[F]))$$

and by Theorem 3.18 this can be transformed into an equivalence

$$K(\text{Mod}_{fp}(R)) \xrightarrow{\sim} K(\text{Mod}_{fp}(R[F])).$$

Combining this with the equivalences in line 3.5 gives our claim in the affine case.

(Step 2) Suppose X is not affine. Two proof ideas come to mind: *(Version I)* The presheaf of spectra $K : U \mapsto K(\text{Coh}(U))$ satisfies Zariski descent. By Corollary 3.11 the same is true for its Frobenius line analogue $KF : U \mapsto K(\text{Coh}_{\mathcal{O}_U[F]}(U))$. Hence, by Zariski descent we may check whether the induced homomorphism

$$K \rightarrow KF, \quad \mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X[F]$$

is an equivalence on affine covers, where it is true by Step 1. *(Version II)* We can also circumvent sheaf methods and follow Quillen's lead in [Qui73, §7.4, Prop. 4.1]. We prove the claim by induction on the dimension of X . If X is zero-dimensional, it is just a collection of finitely many closed points and we are in the affine situation. This case is

already proven. Thus, suppose the case of dimension $n - 1$ is settled and $\dim X = n \geq 1$. Suppose $Z \hookrightarrow X$ is a reduced closed subscheme with $\operatorname{codim}_X Z \geq 1$ and $U := X - Z$ its open complement. Then we have the localization sequence, Proposition 3.10,

$$K(\operatorname{Coh}_{\mathcal{O}_Z[F]}(Z)) \xrightarrow{i_*} K(\operatorname{Coh}_{\mathcal{O}_X[F]}(X)) \xrightarrow{j^*} K(\operatorname{Coh}_{\mathcal{O}_U[F]}(U)) \longrightarrow +1.$$

As K -theory commutes with filtering colimits, we may run over the filtering family of all reduced closed subschemes $Z \hookrightarrow X$ such that $\operatorname{codim}_X Z \geq 1$, ordered by inclusion of the underlying closed subsets in the Zariski topology. We obtain

$$\operatorname{colim}_Z K(\operatorname{Coh}_{\mathcal{O}_Z[F]}) \rightarrow K(\operatorname{Coh}_{\mathcal{O}_X[F]}) \rightarrow \coprod_{x \in X^0} K(\operatorname{Coh}_{\mathcal{O}_{X,x}[F]}) \rightarrow +1.$$

Each entry in the first term agrees with $K(\operatorname{Coh}(Z))$ by our induction hypothesis and the fact that Z has dimension at most $n - 1$. Moreover, as the $x \in X^0$ are generic points, $\dim \mathcal{O}_{X,x} = 0$ and thus $K(\operatorname{Coh}_{\mathcal{O}_{X,x}[F]}) = K(\operatorname{Coh}(\kappa(x)))$, because we are in an affine situation and can use Step 1. Now conclude by the Five Lemma as in Quillen's proof of [Qui73, §7.4, Prop. 4.1]. By induction, the proof is finished. Unravelling the proof of Zariski descent for KF , it is easy to see that both versions of the proof just differ by the geometric intuition employed, yet on the technical level they are entirely equivalent. \square

4. CARTIER MODULES

We recall the basic definitions of the theory of Cartier modules, due to Blickle and Bökke. See [BB11, §2] for details and proofs. Suppose X is a Noetherian scheme, separated over \mathbf{F}_q . We write F for the Frobenius morphism

$$F : X \rightarrow X,$$

which maps f to f^q on the level of the structure sheaves, and is the identity map on the underlying topological space of X . A *coherent Cartier module* (or in alternative terminology: a 'coherent κ -sheaf', see [BB13, §2]) is a coherent \mathcal{O}_X -module sheaf \mathcal{F} along with an \mathcal{O}_X -linear map $C : F_*\mathcal{F} \rightarrow \mathcal{F}$. A morphism of Cartier modules is a morphism $\psi : \mathcal{F} \rightarrow \mathcal{G}$ of the underlying coherent sheaves commuting with the respective maps C , i.e. the diagram

$$\begin{array}{ccc} F_*\mathcal{F} & \xrightarrow{F_*\psi} & F_*\mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{F} & \xrightarrow{\psi} & \mathcal{G} \end{array}$$

is supposed to commute. We write $\operatorname{CohCart}(X)$ for the category of coherent Cartier modules. This is an abelian category. The Cartier module \mathcal{F} is called *nilpotent* if there exists some integer $v \geq 1$ such that the v -th power $C^v : F_*^v \mathcal{F} \rightarrow \mathcal{F}$ is the zero morphism. One can show that \mathcal{F} is nilpotent if and only if this holds for the stalks \mathcal{F}_x at all closed points $x \in X$, [BB11, Lemma 2.10].

The category of nilpotent coherent Cartier modules, call it $\operatorname{CohCart}_{\text{nil}}(X)$, is a Serre subcategory of $\operatorname{CohCart}(X)$, [BB11, Lemma 2.11]. Define the category of *Cartier crystals* as the quotient category

$$\operatorname{CartCrys}(X) := \operatorname{CohCart}(X) / \operatorname{CohCart}_{\text{nil}}(X).$$

Since this is a quotient of an abelian category by a Serre category, this is itself an abelian category. These categories have a very rich theory of pullback and pushforward functors,

both in $*$ - and $!$ -variants. This is a longer story and since we shall not need them, we refer the curious reader to [BB11].

4.1. Riemann–Hilbert correspondence. We shall need a positive characteristic version of the Riemann–Hilbert correspondence: Suppose X/\mathbf{F}_q is a smooth scheme. Let $\mathcal{E}t_c(X, \mathbf{F}_q)$ be the abelian category of constructible étale sheaves with \mathbf{F}_q -coefficients. The derived category $D_c^b(X_{\mathcal{E}t}, \mathbf{F}_q)$ has a perverse t -structure, constructed by Gabber [Gab04], [EK04b, §11.5.2]. Let $\mathcal{E}t_{\text{perv}}(X, \mathbf{F}_q)$ denote the heart of this t -structure. The objects of this category are called *perverse étale sheaves*.

Then there is an equivalence of abelian categories

$$(4.1) \quad \text{Sol} : \text{CartCrys}(X) \xrightarrow{\sim} \mathcal{E}t_{\text{perv}}(X, \mathbf{F}_q)^{op}.$$

This result can be obtained by combining the Riemann–Hilbert correspondence of Emerton–Kisin with a mechanism due to Blickle and Bökke. We give a summary of how this works:

Definition 4.1. *Suppose X/\mathbf{F}_q is regular and F -finite. Then we call it F -dualizing, if the dualizing sheaf ω_X (relative to the base \mathbf{F}_q) is compatible with the Frobenius in the sense of $F^!\omega_X \simeq \omega_X$.*

This condition is discussed in detail in [BB11, §2.4]. Moreover, if X/\mathbf{F}_q is smooth or X is affine, X is automatically F -dualizing, so this condition is harmless. See loc. cit.

We write $\mu_{\text{lfgu}}(X)$ for the abelian category of locally finitely generated unit left $\mathcal{O}_X[F]$ -modules, as introduced by Emerton and Kisin ([EK04b, Definition 6.3]).

Theorem 4.2 (Emerton–Kisin). *Suppose X/\mathbf{F}_q is a smooth scheme. Then there is an equivalence of abelian categories*

$$\mu_{\text{lfgu}}(X) \xrightarrow{\sim} \mathcal{E}t_{\text{perv}}(X, \mathbf{F}_q)^{op}.$$

This is the central result of the first part of [EK04b]. This is also explained in [EK04a].

Theorem 4.3 (Blickle–Bökke [BB11]). *Suppose X/\mathbf{F}_q is a regular, F -finite and F -dualizing scheme. Then there is an equivalence of abelian categories*

$$\text{CartCrys}(X) \xrightarrow{\sim} \mu_{\text{lfgu}}(X).$$

This equivalence is constructed by concatenating a substantial list of other individual equivalences of categories. Let us sketch this: (*Step 1*) There is another category of crystals, γ -crystals, which we denote by $\gamma\text{-Crys}(X)$. It is a quotient category of the γ -sheaf category of [Bli08]. We shall not use it anywhere outside this section. Now, if X/\mathbf{F}_q is regular and F -finite and F -dualizing, then Blickle and Bökke produce an equivalence of abelian categories

$$\begin{aligned} \mathbb{D} : \text{CartCrys}(X) &\longrightarrow \gamma\text{-Crys}(X) \\ \mathcal{F} &\longmapsto \mathcal{F} \otimes_{\mathcal{O}_X} \omega_X^{\vee}, \end{aligned}$$

where ω_X^{\vee} is the tensor inverse of ω_X . See [BB11, Theorem 5.9] for the proof. The functor clearly makes sense, sending Cartier modules to γ -sheaves, and then they show that it factors over the corresponding notions of nilpotent Cartier resp. nilpotent γ -sheaves. Thus, it descends to a functor between the associated quotient categories of

crystals. (*Step 2*) The second step is already explained in Blickle’s paper [Bli08]. Besides introducing γ -sheaves at all, he establishes an equivalence of abelian categories

$$\begin{aligned} \text{Gen} : \gamma\text{-Crys}(X) &\longrightarrow \mu\text{lfgu}(X) \\ \mathcal{F} &\longmapsto \text{colim}(\mathcal{F} \rightarrow F^*\mathcal{F} \rightarrow F^{2*}\mathcal{F} \rightarrow F^{3*}\mathcal{F} \rightarrow \dots), \end{aligned}$$

where the transition morphisms are part of the datum of a γ -sheaf. By [Bli08, Theorem 2.27] such an equivalence exists with ‘minimal γ -sheaves’ on the left-hand side, and by [Bli08, Prop. 3.1] the latter category is equivalent to $\gamma\text{-Crys}(X)$. The above functor is defined as the composition of these equivalences. The essential surjectivity is the existence statement for generators, dating back to the start of the entire business [Lyu97].

Theorem 4.4. *The category of τ -crystals of [BP09], which we will denote by $\tau\text{-Crys}(X)$, is also equivalent to $\text{CartCrys}(X)$.*

This follows by combining the Riemann–Hilbert correspondence of Böckle and Pink [BP09, Theorem 10.3.6] with the above. As far as I understand, a direct construction of this equivalence will appear in [BB].

There is a much bigger panorama involving even more categories and their equivalence and duality relations. We refer to [BB11], [BB13] for the big picture.

Remark 4.5 (Singular case). Recently, the Riemann–Hilbert correspondence in the formulation of line 4.1 has even been shown for singular X in Schedlmeier’s thesis [Sch16], under rather mild hypotheses. It is proven by reduction to the smooth case. Presumably, a part of the following results could be extended to singular X using this.

4.2. K -theory of Cartier crystals. Next, we compute the K -theory of the categories of Cartier modules and Cartier crystals.

Proposition 4.6. *There is a canonical equivalence*

$$K_m(\text{Coh}(X)) \xrightarrow{\sim} K_m(\text{CohCart}_{\text{nil}}(X)).$$

It is induced from the exact functor which sends a coherent sheaf \mathcal{F} to the Cartier module, where the Cartier operator C acts as the zero map.

Proof. It is clear that the relevant functor exists and is exact. Every nilpotent Cartier module can be filtered and its filtered parts have trivial action by F . Thus, each filtered part lies in the essential image of the functor. It follows from Quillen’s dévissage theorem, [Qui73, §5, Theorem 4], that the induced map in K -theory is an equivalence. \square

Proposition 4.7. *There is a canonical long exact sequence*

$$\dots \rightarrow K_m(\text{Coh}(X)) \rightarrow K_m(\text{CohCart}(X)) \xrightarrow{q} K_m(\text{CartCrys}(X)) \rightarrow \dots$$

for all $m \in \mathbf{Z}$.

Proof. We have the exact sequence of abelian categories,

$$\text{CohCart}_{\text{nil}}(X) \hookrightarrow \text{CohCart}(X) \xrightarrow{q} \text{CartCrys}(X).$$

Thus, by Quillen’s localization sequence, [Qui73, §5, Theorem 5], there is a long exact sequence in K -theory groups,

$$\begin{aligned} (4.2) \quad \dots \rightarrow K_m(\text{CohCart}_{\text{nil}}(X)) &\rightarrow K_m(\text{CohCart}(X)) \rightarrow \\ &\dots \xrightarrow{q} K_m(\text{CartCrys}(X)) \rightarrow K_{m-1}(\text{CohCart}_{\text{nil}}(X)) \rightarrow \dots \end{aligned}$$

The result follows from combining this with Prop. 4.6. \square

Definition 4.8. *Being F -finite, the Frobenius is a finite morphism and we get an exact functor of pushforward*

$$F_* : K(\mathrm{Coh}(X)) \longrightarrow K(\mathrm{Coh}(X)).$$

There is an exact forgetful functor

$$U : \mathrm{CohCart}(X) \longrightarrow \mathrm{Coh}(X)$$

which just forgets the right action by F , i.e. it sends a coherent Cartier module just to its underlying coherent \mathcal{O}_X -module. We write “ U ” for “underlying”.

Proposition 4.9. *On the level of K -theory, the inclusion functor*

$$(4.3) \quad \mathrm{CohCart}(X) \longrightarrow \mathrm{Coh}_{\mathcal{O}_X[F]}(X),$$

when composed with the inverse Ψ^{-1} (of the equivalence of Theorem 3.15), i.e.

$$\mathrm{CohCart}(X) \longrightarrow \mathrm{Coh}_{\mathcal{O}_X[F]}(X) \xleftarrow[\Psi]{\sim} \mathrm{Coh}(X),$$

agrees with $(1 - F_) \circ U$.*

Proof. Given any coherent Cartier module \mathcal{F} , we have an exact sequence of right $\mathcal{O}_X[F]$ -modules,

$$0 \longrightarrow \tilde{\mathcal{F}}[X] \longrightarrow \mathcal{F}[X] \longrightarrow \mathcal{F} \longrightarrow 0,$$

by using the sheaf version of Lemma 2.21. By Lemma 2.22 this is a short exact sequence in the category $\mathrm{Coh}_{\mathcal{O}_X[F]}(X)$. We get three functors $s_i : \mathrm{CohCart}(X) \rightarrow \mathrm{Coh}_{\mathcal{O}_X[F]}(X)$ for $i = 1, 2, 3$:

$$\mathcal{F} \longrightarrow \tilde{\mathcal{F}}[X], \quad \mathcal{F} \longrightarrow \mathcal{F}[X], \quad \mathcal{F} \longrightarrow \mathcal{F}.$$

The last functor is obviously exact. The middle functor can also be written as $\mathcal{F}[X] = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X[F]$ and since $\mathcal{O}_X[F]$ is a free and thus flat left \mathcal{O}_X -module sheaf, this functor is also exact. The leftmost functor arises analogously, intertwined with the exact pushforward F_* , so it is also an exact functor. By the Additivity Theorem [Qui73, §3], it follows that $s_{2*} = s_{1*} + s_{3*}$ on K -theory. We note that $s_{2*} = \Psi \circ U$ (by the very definition; recall the explicit functor of Theorem 3.15. Note that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X[F]$ only depends on the (right) \mathcal{O}_X -module structure of \mathcal{F} , so we tacitly have forgotten its right-action by F) and by Remark 2.11, $s_{1*} = \Psi \circ F_* \circ U$. Thus, $s_{3*} = \Psi \circ (1 - F_*) \circ U$, but s_{3*} is just the inclusion of line 4.3. This proves the claim. \square

Next, we recall the following classical result in algebra:

Theorem 4.10 (Wedderburn’s Theorem). *Every finite division ring is a field.*

Proposition 4.11. *Suppose X is F -finite.*

- (1) *For $m \geq 1$ the group $K_m(\mathrm{CartCrys}(X))$ is a pure torsion group, of order prime to p . In particular,*

$$K_m(\mathrm{CartCrys}(X)) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)} = 0 \quad \text{for} \quad m \geq 1,$$

- (2) *$K_m(\mathrm{CartCrys}(X)) = 0$ for $m = 2i$, $i > 1$, and*
- (3) *$K_0(\mathrm{CartCrys}(X))$ is the free abelian group whose generators correspond to the iso-classes of simple Cartier crystals.*

Proof. Firstly, we use that if X is F -finite, then the abelian category $\mathbf{CartCrys}(X)$ is both Noetherian and Artinian by [BB11, Corollary 4.7]. In particular, every object in it has finite length. Let $\mathbf{CartCrys}(X)^{ss}$ be the full subcategory of simple objects, i.e. objects which do not admit any non-trivial subobjects. By Quillen's dévissage theorem, [Qui73, §5, Theorem 4], the inclusion functor $i : \mathbf{CartCrys}(X)^{ss} \hookrightarrow \mathbf{CartCrys}(X)$ induces an equivalence of K -groups,

$$(4.4) \quad K_m(\mathbf{CartCrys}(X)^{ss}) \xrightarrow{\sim} K_m(\mathbf{CartCrys}(X))$$

for all $m \in \mathbf{Z}$. Being semisimple abelian, we have an equivalence of \mathbf{F}_q -linear abelian categories

$$(4.5) \quad \mathbf{CartCrys}(X)^{ss} \longrightarrow \coprod_Z \mathbf{Mod}(D_Z),$$

where Z runs through the simple objects, and $D_Z := \mathbf{End}_{\mathbf{CartCrys}}(Z)$ denotes their endomorphism algebras. These must be finite fields by [BB11, Corollary 4.16] – in detail: Being simple, Schur's Lemma implies that D_Z is a division ring over \mathbf{F}_q . Coherence of Z and reduction to the single underlying associated prime implies that D_Z is also a finite-dimensional \mathbf{F}_q -vector space. Thus, by Wedderburn's Theorem, the finite division ring D_Z must be a (necessarily finite) field. We obtain

$$D_Z \simeq \mathbf{F}_{p^{r(Z)}} \quad (\text{for } r(Z) \geq 1 \text{ appropriately chosen depending on } Z)$$

and therefore line 4.5 implies that

$$K_m(\mathbf{CartCrys}(X)^{ss}) \cong \coprod_Z K_m(D_Z) = \coprod_Z K_m(\mathbf{F}_{p^{r(Z)}})$$

and by Quillen's computation of the K -theory of finite fields [Qui72],

$$K_m(\mathbf{F}_{p^r}) = \begin{cases} \mathbf{Z}/(p^{ri} - 1) & \text{for } m \geq 1 \text{ odd, } m = 2i - 1 \\ 0 & \text{for } m \geq 1 \text{ even.} \end{cases}$$

it follows that for $m \geq 1$ the group $K_m(\mathbf{CartCrys}(X)^{ss})$ is a direct sum of prime-to- p pure torsion groups, and vanishes for even $m \geq 2$. Moreover, K_0 of any field is \mathbf{Z} . Along with the isomorphism in line 4.4 this yields our claim. \square

A different description of the K_0 -group has been developed by Taelman in [Tae15]: He describes it by his version of the function-sheaf correspondence. Let $X(\mathbf{F}_q)$ denote the set of \mathbf{F}_q -rational points of X , and $\mathbf{Map}(X(\mathbf{F}_q), \mathbf{F}_q)$ the vector space of set-theoretic maps (i.e. literally assigning an element of \mathbf{F}_q to each \mathbf{F}_q -rational point).

Theorem 4.12 (Taelman, [Tae15, Theorem 3.6]). *Suppose X/\mathbf{F}_q is a finite type scheme. Then there is a short exact sequence of abelian groups*

$$0 \longrightarrow R \longrightarrow K_0(\mathbf{CartCrys}(X)) \xrightarrow{\text{tr}} \mathbf{Map}(X(\mathbf{F}_q), \mathbf{F}_q) \longrightarrow 0,$$

given by the function-sheaf correspondence of [Tae15, Ch. 1, §2]. The group R is the subgroup generated by the differences $[(\mathcal{F}, c)] + [(\mathcal{F}, c')] - [(\mathcal{F}, c + c')]$, where c, c' specify the action of the Cartier operators. The group R contains all p -th multiples, i.e. $pK_0(\mathbf{CartCrys}(X))$.

Loc. cit. this result is phrased for the K_0 -group of the category of τ -crystals. However, this category is equivalent to $\text{CartCrys}(X)$ by Böckle and Pink (Theorem 4.4). Let us temporarily remain in the context of τ -sheaves: Taelman also shows that for every \mathbf{F}_q -rational point $x : \text{Spec } \mathbf{F}_q \hookrightarrow X$ the pushforward (in the theory of τ -sheaves) of $x_* \mathbf{1}$, is a simple τ -crystal such that

$$\text{tr}(x_* \mathbf{1})(y) = \delta_{y=x}$$

for $y \in X(\mathbf{F}_q)$. In other words, it is a canonical preimage under his function-sheaf correspondence of the delta function supported exclusively at the given point x . By the equivalence of the categories of τ -crystals and Cartier crystals, the computation $K_0(\dot{\text{Et}}_c(X, \mathbf{F}_q)) = \bigoplus \mathbf{Z}$ can also be performed in the context of τ -crystals. The $\#X(\mathbf{F}_q)$ pairwise non-isomorphic τ -crystals $x_* \mathbf{1}$ then provide a canonical subset of the simple objects.

We arrive at one of our main results:

Theorem 4.13. *Suppose X/\mathbf{F}_q is a smooth scheme. Then:*

(1) *There is a canonical equivalence*

$$K(\text{CartCrys}(X)) \xrightarrow{\sim} K(\dot{\text{Et}}_c(X, \mathbf{F}_q)).$$

(2) *Moreover,*

$$K_m(\dot{\text{Et}}_c(X, \mathbf{F}_q)) = \begin{cases} \text{prime-to-}p \text{ torsion} & \text{for } m = 2i + 1, \\ 0 & \text{for } m = 2i, i > 0, \\ \bigoplus \mathbf{Z} & \text{for } m = 0, \end{cases}$$

where $i \geq 0$, and the direct sum in the last row runs over all simple objects of $\text{CartCrys}(X)$, or equivalently perverse sheaves $\dot{\text{Et}}_{\text{perv}}(X, \mathbf{F}_q)$. Among them, there is a canonical set of $\#X(\mathbf{F}_q)$ generators which surject on the right-hand side in Sequence 0.3, while the remaining generators all map to zero under the function-sheaf correspondence.

Proof. (1) By the Riemann–Hilbert correspondence of Emerton and Kisin [EK04b], in the concrete shape of Equation 4.1, we have an equivalence

$$K(\text{CartCrys}(X)) \xrightarrow{\sim} K(\dot{\text{Et}}_{\text{perv}}(X, \mathbf{F}_q))$$

since K -theory is not affected by switching to the opposite category. On the right-hand side the perverse t -structure plays the essential rôle on the level of categories. However, on the level of K -theory this washes out: Thanks to Neeman’s Theorem of the Heart (see the survey [Nee05, Theorem 50] and the following discussion; or [Bar15]) the following holds: Suppose \mathbf{T} is a triangulated category which admits a Waldhausen model, and \mathbf{A} and \mathbf{B} two abelian categories which arise as hearts of two bounded t -structures on \mathbf{T} , then there is an equivalence in K -theory, $K(\mathbf{A}) \xrightarrow{\sim} K(\mathbf{B})$. Bounded complexes of étale \mathbf{F}_q -sheaves with constructible cohomology, $C_c^b(X_{\text{ét}}, \mathbf{F}_q)$, provide a dg and Waldhausen model for $D_c^b(X_{\text{ét}}, \mathbf{F}_q)$. We deduce that

$$(4.6) \quad K(\dot{\text{Et}}_c(X, \mathbf{F}_q)) \xrightarrow{\sim} K(\dot{\text{Et}}_{\text{perv}}(X, \mathbf{F}_q))$$

by Neeman’s theorem. The reader can find an analogous procedure explained and carried out in [Bar15, §7], where it is applied to the perverse t -structure on coherent sheaves.

(2) The discussion of this section implies the claims for $K(\text{CartCrys}(X))$, namely Prop. 4.11 and Taelman’s result, and then use part (1) of the proof. \square

Note that the comparison of line 4.6 only works on the level of K -theory. As abelian categories, $\mathcal{E}t_{\text{perv}}(X, \mathbf{F}_q)$ and $\mathcal{E}t_c(X, \mathbf{F}_q)$ are very different.

5. QUILLEN'S COMPUTATION – REVISITED

In “Higher Algebraic K -Theory I” [Qui73] Quillen studies not only the K -theory of schemes, but also some examples of non-commutative rings. The prominent and better known example are central simple algebras A and his computation

$$K(X) \xrightarrow{\sim} \coprod K(A^{\otimes i}),$$

where X is the associated Severi–Brauer variety. This computation is, in a way, a “slightly non-commutative” generalization of the computation of the K -theory of projective space \mathbf{P}_k^n .

Instead, we shall focus on the other, less well-known, example. It was a motivation for this paper, because one runs into severe technical problems when trying to generalize it, and unlike the above, it is far less clear how to connect it to “geometry”. Quillen gives this example on the pages⁶ 38–39: He fixes a prime power $q = p^r$, $r \geq 1$, defines $k := \mathbf{F}_q^{\text{sep}}$ and considers the Frobenius skew ring $A := k[F]$ over k . Then he defines D to be the quotient skew field of A ,

$$(5.1) \quad D := \text{Quot } A.$$

Quillen now computes the K -theory of this skew field:

$$(5.2) \quad K_m(D) = \begin{cases} \mathbf{Z} & \text{if } m = 0 \\ \mathbf{Z} \oplus \mathbf{Z} & \text{if } m = 1 \\ (K_{2i-1}\mathbf{F}_q)^{\oplus 2} & \text{if } m = \text{even} \\ 0 & \text{if } m = \text{odd}. \end{cases}$$

We will now adapt this computation to smooth projective schemes over \mathbf{F}_q : We first need to define D in general, for example for a general ring R instead of $\mathbf{F}_q^{\text{sep}}$. We should recall its definition: For an associative unital domain R , the set of non-zero elements may satisfy the Ore conditions. If this is the case, it is called a *left* (res. *right*) *Ore domain*. Then the localization $D := (R - \{0\})^{-1}R$ is a skew field [Coh95, Cor. 1.3.3]⁷. Now, the following characterization holds:

Lemma 5.1. *Suppose k/\mathbf{F}_q is some field extension. Then the Frobenius skew ring $k[F]$ is a left (or right) Ore domain if and only if k is perfect.*

Proof. [Coh95, Prop. 2.1.6]. Use that $k[F]$ is an example of a twisted polynomial ring for the endomorphism σ being the Frobenius $x \mapsto x^q$, and the trivial derivation $\delta := 0$. \square

Remark 5.2. There is a relation between the lack of a field of fractions and the lack of Noetherianity. Every right Noetherian domain is a right Ore domain, [MR01, Ch. 2, (1.15) Theorem] or [Lam99, Ch. 4, (10.23) Corollary]. This is a part of Yoshino’s result, Thm. 2.1.

⁶or equivalently pages 114–115, 122–123, depending on which of the three incompatible paginations of Quillen’s paper the reader wishes to use

⁷Note that Cohn writes “field” for what we would call a “skew field” and “ R^\times ” for $R - \{0\}$, while we would write R^\times to denote the group of units of the ring R .

In particular, in the context of Quillen's paper, where k is algebraically closed and thus trivially perfect, this quotient skew field exists and he can work with it. However, we run into a serious problem when trying to understand the broader picture of his computation since it becomes unclear how the definition of D is to be generalized. For example, it is hopeless to try to define such a D for a curve X/\mathbf{F}_q if already for its function field $\mathbf{F}_q(X)$ the multiplicative set $\mathbf{F}_q(X) - \{0\}$ is not a left or right denominator set. However, Quillen goes on to characterize the finitely generated right D -modules as

$$(5.3) \quad \text{Mod}_{fg}(D) = \text{Mod}_{fg}(A)/\mathbf{B},$$

where \mathbf{B} is the Serre subcategory of those A -modules which are finite-dimensional as k -vector spaces. So, instead of worrying about defining D , we can generalize its category of modules. The analogue of $\text{Mod}_{fg}(A)$ will be $\text{Coh}_{\mathcal{O}_X[F]}(X)$ and \mathbf{B} have to be those modules which are finitely generated over \mathcal{O}_X , but this is precisely the subcategory of Cartier modules in the sense of Bickel and Böckle [BB11]:

Definition 5.3. *If X/\mathbf{F}_q is an F -finite Noetherian separated scheme, define an abelian category⁸*

$$(5.4) \quad \text{QD}(X) := \text{Coh}_{\mathcal{O}_X[F]}(X)/\text{CohCart}(X).$$

This definition is justified by the following:

Lemma 5.4. *If X/\mathbf{F}_q is an F -finite Noetherian separated scheme, then $\text{CohCart}(X)$ is a Serre subcategory of $\text{Coh}_{\mathcal{O}_X[F]}(X)$.*

Proof. First of all, it is a full subcategory: Cartier modules carry, by definition, a right $\mathcal{O}_X[F]$ -module structure. So it only remains to show that being coherent as an \mathcal{O}_X -module sheaf implies being finitely presented as a right $\mathcal{O}_X[F]$ -module sheaf. This can be checked affine locally, and then reduces to Lemma 2.22. Secondly, as X is Noetherian, the condition to be finitely generated over \mathcal{O}_X renders this full subcategory a Serre subcategory. \square

Example 5.5. Let us return to Quillen's original example. Clearly $X := \text{Spec } \mathbf{F}_q^{\text{sep}}$ is an F -finite Noetherian scheme, its sheaves identify with certain types of $\mathbf{F}_q^{\text{sep}}$ -modules, notably $\text{Coh}_{\mathcal{O}_X[F]}(X) \cong \text{Mod}_{fg}(A)$ and $\text{CohCart}(X) \cong \mathbf{B}$ in Quillen's notation. In particular,

$$\text{QD}(X) \cong \text{Mod}_{fg}(D).$$

Quillen then proceeds to compute the K -theory of the skew field D using his localization sequence, based on modules over D being a quotient abelian category, as in Equation 5.3. We can do the same for $\text{QD}(X)$, and for the same reason.

Theorem 5.6. *Suppose X/\mathbf{F}_q is a regular and F -finite Noetherian separated scheme. Then there is a long exact sequence*

$$\cdots \longrightarrow K_m(\text{CohCart}(X)) \xrightarrow{(*)} K_m(X) \xrightarrow{\overline{\Psi}} K_m(\text{QD}(X)) \longrightarrow \cdots,$$

where the map $(*)$ is $(1 - F_*) \circ U$.

⁸I chose $\text{QD}(X)$ because $\text{D}(X)$ looks too much as if it were to suggest a derived category.

Proof. By Quillen's localization theorem, we have the homotopy fiber sequence

$$K(\mathrm{CohCart}(X)) \xrightarrow{\iota} K(\mathrm{Coh}_{\mathcal{O}_X[F]}(X)) \xrightarrow{\overline{(-)}} K(\mathrm{QD}(X)) \longrightarrow +1,$$

straight from Definition 5.3, where ι is the inclusion of the Serre subcategory and $\overline{(-)}$ denotes the exact functor to the quotient abelian category. We modify this sequence in two ways: (1) By Theorem 3.15 we have the Frobenius analogue of the \mathbf{A}^1 -invariance, $\Psi : K(\mathrm{Coh}(X)) \xrightarrow{\sim} K(\mathrm{Coh}_{\mathcal{O}_X[F]}(X))$. We arrive at

$$(5.5) \quad K(\mathrm{CohCart}(X)) \xrightarrow{\Psi^{-1} \circ \iota} K(\mathrm{Coh}(X)) \xrightarrow{\overline{(-)} \circ \Psi} K(\mathrm{QD}(X)) \longrightarrow +1.$$

By Prop. 4.9 the first arrow agrees with $(1 - F_*) \circ U$. \square

Remark 5.7. The fundamental rôle of the homotopy fiber of $1 - F_*$ has its counterpart in [Gra88], where $1 - F^*$ has a similar function.

5.1. Implications of Parshin's conjecture.

Conjecture 1 (Parshin). *Suppose X/\mathbf{F}_q is a smooth projective scheme. Then $K_m(X) \otimes \mathbf{Q} = 0$ for $m \geq 1$.*

To the best of my knowledge, the first mention of this in print is [Bei84, Conj. 2.4.2.3]. At present, this conjecture is only known for curves, or when the K -theory is basically entirely known anyway, e.g. for cellular varieties. Next, let us recall Quillen's computation, Equation 5.2, but with rational coefficients. This means, we are talking about

$$K_m(D)_{\mathbf{Q}} = \begin{cases} \mathbf{Q} & \text{if } m = 0 \\ \mathbf{Q} \oplus \mathbf{Q} & \text{if } m = 1, \\ 0 & \text{if } m \geq 2. \end{cases}$$

If we assume the validity of Parshin's Conjecture, we can see the characteristic features of this computation repeat, in a more complicated fashion, in the general case of varieties:

Theorem 5.8. *Suppose X/\mathbf{F}_q is a smooth, projective, and geometrically integral scheme. Then*

$$K_m(\mathrm{QD}(X))_{\mathbf{Q}} = \begin{cases} \mathbf{Q} & \text{if } m = 0 \\ \mathbf{Q} \oplus K_0(\dot{\mathrm{Et}}_c(X, \mathbf{F}_q))_{\mathbf{Q}} & \text{if } m = 1, (*) \\ 0 & \text{if } m \geq 2, (*) \end{cases}$$

where the entries marked with an asterisk $(*)$ are only known if X satisfies Parshin's Conjecture. Unconditionally: If $x : \mathrm{Spec} \kappa(x) \hookrightarrow X$ is any closed point, $[(x_* \mathcal{O}_{\kappa(x)}) \otimes_{\mathcal{O}_X} \mathcal{O}_X[F]]$ is a basis of $K_0(\mathrm{QD}(X))_{\mathbf{Q}}$.

Proof. We use Theorem 5.6. (1) For $m = 0$, we obtain the presentation

$$K_0(\mathrm{CohCart}(X))_{\mathbf{Q}} \rightarrow K_0(\mathrm{Coh}(X))_{\mathbf{Q}} \rightarrow K_0(\mathrm{QD}(X))_{\mathbf{Q}} \rightarrow 0,$$

and we know that the first arrow takes the value $(1 - F_*)U[\mathcal{F}]$ for Cartier modules \mathcal{F} . Since U maps \mathcal{F} to the underlying coherent sheaf, and all coherent sheaves can be equipped with the trivial Cartier module structure of F acting by zero, the image is just $(1 - F_*)$ applied to arbitrary coherent sheaves. By this and the regularity of X , we can also work with the right exact sequence

$$K_0(X)_{\mathbf{Q}} \xrightarrow{1-F_*} K_0(X)_{\mathbf{Q}} \rightarrow K_0(\mathrm{QD}(X))_{\mathbf{Q}} \rightarrow 0.$$

Using Grothendieck-Riemann-Roch, $K_0(X) \otimes \mathbf{Q} \cong \coprod_{i \geq 0} \mathrm{CH}_i(X)_{\mathbf{Q}}$, and the Frobenius pushforward acts as q^i on $\mathrm{CH}_i(X)_{\mathbf{Q}}$ (by direct computation, or by [Sou84, §1.5.3, Prop. 2]). For $i \geq 1$, the element $1 - q^i$ is invertible in the rationals, so $1 - F_*$ acts as an isomorphism on these summands, while it is the zero map for $\mathrm{CH}_0(X)_{\mathbf{Q}}$. Thus, we get an isomorphism

$$(5.6) \quad \mathrm{CH}_0(X)_{\mathbf{Q}} \xrightarrow{\cong} K_0(\mathrm{QD}(X)) \otimes \mathbf{Q}.$$

The zero cycles sit in an exact sequence, $0 \rightarrow A_0(X) \rightarrow \mathrm{CH}_0(X) \rightarrow \mathbf{Z} \rightarrow 0$, with the degree map. By Kato-Saito unramified class field theory, specifically [KS83, Theorem 1], the reciprocity map sends this sequence to $0 \rightarrow \pi_1^{\mathrm{geom}}(X, *)_{ab} \rightarrow \pi_1^{\mathrm{et}}(X, *)_{ab} \rightarrow \mathrm{Gal}(\mathbf{F}_q^{\mathrm{sep}}/\mathbf{F}_q) \rightarrow 0$ such that it induces an isomorphism on the left term, and the profinite completion $\mathbf{Z} \hookrightarrow \widehat{\mathbf{Z}}$ on the right. Moreover, by Katz-Lang Finiteness [KL81, Theorem 2], applied to the structural morphism $X \rightarrow \mathrm{Spec} \mathbf{F}_q$, the geometric part $\pi_1^{\mathrm{geom}}(X, *)_{ab}$ is finite (the field \mathbf{F}_q is clearly accessible in the sense loc. cit. since it is even finite over its prime field). Hence, line 5.6 implies $\mathrm{CH}_0(X)_{\mathbf{Q}} \cong \mathbf{Q}$ and that $K_0(\mathrm{QD}(X)) \otimes \mathbf{Q}$ is one-dimensional. Finally, any closed point generates $\mathrm{CH}_0(X)_{\mathbf{Q}}$ and unwinding the maps, this gives the explicit generator $[(x_* \mathcal{O}_{\mathbf{F}_q}) \otimes_{\mathcal{O}_X} \mathcal{O}_X[F]]$.

(2) For $m = 1$, we get

$$(5.7) \quad \cdots \rightarrow K_1(X)_{\mathbf{Q}} \rightarrow K_1(\mathrm{QD}(X))_{\mathbf{Q}} \rightarrow K_0(\mathrm{CohCart}(X))_{\mathbf{Q}} \xrightarrow{(*)} K_0(X)_{\mathbf{Q}}.$$

By Parshin's Conjecture, $K_1(X)_{\mathbf{Q}} = 0$, so it follows that $K_1(\mathrm{QD}(X))_{\mathbf{Q}}$ is the kernel of the morphism $(*)$. We compute this kernel as follows: By Prop. 4.7 we have the exact sequence

$$K_1(\mathrm{CartCrys}(X)) \rightarrow K_0(\mathrm{Coh}(X)) \rightarrow K_0(\mathrm{CohCart}(X)) \rightarrow K_0(\mathrm{CartCrys}(X)) \rightarrow 0$$

and by Prop. 4.11 the group $K_1(\mathrm{CartCrys}(X))$ is pure torsion. We obtain the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & K_0(\mathrm{Coh}(X))_{\mathbf{Q}} & \rightarrow & K_0(\mathrm{CohCart}(X))_{\mathbf{Q}} & \rightarrow & K_0(\mathrm{CartCrys}(X))_{\mathbf{Q}} \rightarrow 0 \\ & & \downarrow 1-F_* & & \downarrow (1-F_*)U & & \downarrow \\ 0 & \rightarrow & K_0(\mathrm{Coh}(X))_{\mathbf{Q}} & \xrightarrow{\sim} & K_0(\mathrm{Coh}(X))_{\mathbf{Q}} & \longrightarrow & 0 \longrightarrow 0 \end{array}$$

and since $1 - F_*$ acts as $1 - q^i$ on the rationalized CH_i summand, the left and middle downward arrow have $\mathrm{CH}_0(X)_{\mathbf{Q}}$ both as kernel and cokernel. The snake lemma yields a long exact sequence

$$0 \rightarrow \mathrm{CH}_0(X)_{\mathbf{Q}} \rightarrow K_1(\mathrm{QD}(X))_{\mathbf{Q}} \rightarrow K_0(\mathrm{CartCrys}(X))_{\mathbf{Q}} \rightarrow \mathrm{CH}_0(X)_{\mathbf{Q}} \xrightarrow{\sim} \mathrm{CH}_0(X)_{\mathbf{Q}} \rightarrow 0$$

and the last arrow is an isomorphism since both downward arrows have the same image, so the snake map must be the zero map. Finally, use the Riemann-Hilbert correspondence to identify the K -theory of the Cartier crystals with the one of $\widehat{\mathrm{Et}}_c$ (Theorem 4.13).

(3) For $m \geq 2$, Theorem 5.6 tells us that

$$(5.8) \quad \cdots \rightarrow K_m(X)_{\mathbf{Q}} \rightarrow K_m(\mathrm{QD}(X))_{\mathbf{Q}} \rightarrow K_{m-1}(\mathrm{CohCart}(X))_{\mathbf{Q}} \rightarrow \cdots$$

is exact. By Parshin's Conjecture, the rationalized K -groups $K_m(X)_{\mathbf{Q}}$ vanish. Moreover, in

$$K_{m-1}(\mathrm{Coh}(X))_{\mathbf{Q}} \rightarrow K_{m-1}(\mathrm{CohCart}(X))_{\mathbf{Q}} \rightarrow K_{m-1}(\mathrm{CartCrys}(X))_{\mathbf{Q}}$$

of Prop. 4.7 the left term vanishes by Parshin's Conjecture, and the right one by Prop. 4.11. Thus, $K_{m-1}(\mathrm{CohCart}(X))_{\mathbf{Q}} = 0$ and thus $K_m(\mathrm{QD}(X))_{\mathbf{Q}} = 0$ by the exactness of line 5.8. This finishes the proof. \square

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REFERENCES

- [Bar15] C. Barwick, *On exact ∞ -categories and the theorem of the heart*, Compos. Math. **151** (2015), no. 11, 2160–2186. MR 3427577 4.2, 4.2
- [BB] M. Blickle and G. Böckle, *Cartier crystals: Duality*, forthcoming. 4.1
- [BB11] ———, *Cartier modules: finiteness results*, J. Reine Angew. Math. **661** (2011), 85–123. MR 2863904 (document), 3.2, 4, 4.1, 4.3, 4.1, 4.1, 4.2, 4.2, 5
- [BB13] M. Blickle and G. Böckle, *Cartier crystals*, arXiv:1309.1035 [math.AG], 2013. 4, 4.1
- [Bei84] A. Beilinson, *Higher regulators and values of L-functions*, Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, pp. 181–238. MR 760999 5.1
- [BG73] K. Brown and S. Gersten, *Algebraic K-theory as generalized sheaf cohomology*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 266–292. Lecture Notes in Math., Vol. 341. MR 0347943 3.2
- [BGT16] A. Blumberg, D. Gepner, and G. Tabuada, *K-theory of endomorphisms via noncommutative motives*, Trans. Amer. Math. Soc. **368** (2016), no. 2, 1435–1465. MR 3430369 (document)
- [Bli08] M. Blickle, *Minimal γ -sheaves*, Algebra Number Theory **2** (2008), no. 3, 347–368. MR 2407119 4.1
- [BP09] G. Böckle and R. Pink, *Cohomological theory of crystals over function fields*, EMS Tracts in Mathematics, vol. 9, European Mathematical Society (EMS), Zürich, 2009. MR 2561048 4.4, 4.1
- [Coh91] P. M. Cohn, *Algebra. Vol. 3*, second ed., John Wiley & Sons, Ltd., Chichester, 1991. MR 1098018 3.1
- [Coh95] ———, *Skew fields*, Encyclopedia of Mathematics and its Applications, vol. 57, Cambridge University Press, Cambridge, 1995, Theory of general division rings. MR 1349108 3.1, 5, 5
- [EK04a] M. Emerton and M. Kisin, *An introduction to the Riemann-Hilbert correspondence for unit F-crystals*, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 677–700. MR 2099082 4.1
- [EK04b] ———, *The Riemann-Hilbert correspondence for unit F-crystals*, Astérisque (2004), no. 293, vi+257. MR 2071510 (document), 3.4, 4.1, 4.1, 4.1, 4.2
- [Eme08] M. Emerton, *On a class of coherent rings, with applications to the smooth representation theory of GL_2 in characteristic p* , unpublished, <http://www.math.uchicago.edu/emerton/preprints.html> (2008). (document), 4, 2.7, 2.2, 2.2
- [Gab62] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448. MR 0232821 3.9, 3.2, 3.2
- [Gab04] O. Gabber, *Notes on some t-structures*, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 711–734. MR 2099084 4.1
- [Ger74] S. M. Gersten, *K-theory of free rings*, Comm. Algebra **1** (1974), 39–64. MR 0396671 4, 2.6, 2.2, 3.3, 3.17, 3.18
- [Gra77] D. Grayson, *The K-theory of endomorphisms*, J. Algebra **48** (1977), no. 2, 439–446. MR 0480699 (document)
- [Gra79] ———, *K_2 and the K-theory of automorphisms*, J. Algebra **58** (1979), no. 1, 12–30. MR 535839 (document)
- [Gra80] ———, *K-theory and localization of noncommutative rings*, J. Pure Appl. Algebra **18** (1980), no. 2, 125–127. MR 585217 3.2.1

- [Gra88] ———, *The K-theory of semilinear endomorphisms*, J. Algebra **113** (1988), no. 2, 358–372. MR 929766 (document), 3.16, 5.7
- [Gra95] ———, *Weight filtrations via commuting automorphisms*, K-Theory **9** (1995), no. 2, 139–172. MR 1340843 (document)
- [Jar04] J. F. Jardine, *Generalised sheaf cohomology theories*, Axiomatic, enriched and motivic homotopy theory, NATO Sci. Ser. II Math. Phys. Chem., vol. 131, Kluwer Acad. Publ., Dordrecht, 2004, pp. 29–68. MR 2061851 3.11
- [KL81] N. Katz and S. Lang, *Finiteness theorems in geometric classfield theory*, Enseign. Math. (2) **27** (1981), no. 3-4, 285–319 (1982), With an appendix by Kenneth A. Ribet. MR 659153 5.1
- [KS83] K. Kato and S. Saito, *Unramified class field theory of arithmetical surfaces*, Ann. of Math. (2) **118** (1983), no. 2, 241–275. MR 717824 5.1
- [Kun69] E. Kunz, *Characterizations of regular local rings for characteristic p* , Amer. J. Math. **91** (1969), 772–784. MR 0252389 2
- [Lam99] T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999. MR 1653294 (99i:16001) 2.3, 3.1, 3.3, 5.2
- [Lyu97] G. Lyubeznik, *F-modules: applications to local cohomology and D-modules in characteristic $p > 0$* , J. Reine Angew. Math. **491** (1997), 65–130. MR 1476089 4.1
- [Ma14] L. Ma, *The category of F-modules has finite global dimension*, J. Algebra **402** (2014), 1–20. MR 3160413 3, 2.3, 2.3
- [MR01] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, revised ed., Graduate Studies in Mathematics, vol. 30, American Mathematical Society, Providence, RI, 2001, With the cooperation of L. W. Small. MR 1811901 2.1, 5.2
- [Nee05] A. Neeman, *The K-theory of triangulated categories*, Handbook of K-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 1011–1078. MR 2181838 4.2
- [Qui72] D. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. (2) **96** (1972), 552–586. MR 0315016 4.2
- [Qui73] ———, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341. MR 0338129 (49 #2895) (document), 1.1, 3.2, 3.2, 3.3, 3.3, 4.2, 4.2, 4.2, 4.2, 5
- [Sch16] T. Schedlmeier, *Cartier crystals and perverse constructible étale p -torsion sheaves*, arXiv:1603.07696 [math.AG] (2016). 4.5
- [Sou70] J.-P. Soublin, *Anneaux et modules cohérents*, J. Algebra **15** (1970), 455–472. MR 0260799 4, 2.1, 2.2, 1, 2, 2.5, 2.2
- [Sou84] C. Soulé, *Groupes de Chow et K-théorie de variétés sur un corps fini*, Math. Ann. **268** (1984), no. 3, 317–345. MR 751733 5.1
- [Sou04] ———, *Les variétés sur le corps à un élément*, Mosc. Math. J. **4** (2004), no. 1, 217–244, 312. MR 2074990 3.16
- [Sta16] Stacks, *Stacks Project*, <http://stacks.math.columbia.edu>, 2016. 3.2
- [Tab14] G. Tabuada, *Witt vectors and K-theory of automorphisms via noncommutative motives*, Manuscripta Math. **143** (2014), no. 3-4, 473–482. MR 3167624 (document)
- [Tae15] L. Taelman, *Sheaves and functions modulo p* , Cambridge University Press, 2015. (document), 4.2, 4.12
- [Wei13] C. Weibel, *The K-book*, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, RI, 2013, An introduction to algebraic K-theory. MR 3076731 3.2
- [WY92] C. Weibel and D. Yao, *Localization for the K-theory of noncommutative rings*, Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989), Contemp. Math., vol. 126, Amer. Math. Soc., Providence, RI, 1992, pp. 219–230. MR 1156514 3.2.1
- [Yos94] Y. Yoshino, *Skew-polynomial rings of Frobenius type and the theory of tight closure*, Comm. Algebra **22** (1994), no. 7, 2473–2502. MR 1271618 2.1, 2.1, 3.1, 3.2

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